Polynomial Heisenberg algebras

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Abstract. Polynomial deformations of the Heisenberg algebra are studied in detail. Natural realizations of them are given by the higher order susy partners of the harmonic oscillator for even order polynomials. Here, it is shown that the susy partners of the radial oscillator play a similar role when the order of the polynomial is odd. Indeed, it will be proved that the general systems ruled by such a kind of algebras, in the quadratic and cubic cases, involve Painlevé transcendents of type IV and V, respectively.

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1. Introduction

Deformations of the standard Lie algebras play an important role in diverse interesting problems in physics. We can mention just a couple of examples: the Higgs algebra [1] and the applications to some exactly-solvable Hamiltonians [2]. In these structures, some of the commutators, which in the Lie case are linear combinations of the generators, are replaced by certain non-linear functions [3, 4]. If the Lie algebra is associated to an initial Hamiltonian, the deformed Lie algebra will lead to another Hamiltonian whose spectrum will be a certain variant of the original one.

In this paper we are interested in polynomial Heisenberg algebras, i.e., we will study systems for which the commutators of the Hamiltonian \( H \) with the annihilation and creation (ladder) operators \( L^\pm \) are the same as for the harmonic oscillator, but the commutator of \( L^- \) and \( L^+ \) is a \( m \)-th order polynomial \( P_m(H) \) in \( H \). Concrete realizations of these polynomial algebras are built by taking \( L^\pm \) as \((m + 1)\)-th differential operators [5, 6, 7, 8, 9, 10, 11, 12]. For example, the higher order supersymmetric (hsusy) partners of the harmonic oscillator provide such realizations for even values of \( m \) [11, 12]. Here, we will show that the hsusy partners of the radial oscillator do the same when \( m \) is odd.

Another important question is to study, not just particular realizations, but the characterization of the most general systems ruled by polynomial Heisenberg algebras. It will be seen that the difficulty involved in this problem grows up when increasing the order of the polynomial: for \( m = 0 \) and \( m = 1 \) (linear case) these systems are precisely the harmonic

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and radial oscillators, respectively [5, 6, 7, 9]. We will analyse the next step by showing that, for \( m = 2 \) and \( m = 3 \), the determination of the potentials require to solve Painlevé equations of type IV and V, respectively [7, 8, 13]. Further steps, in which the presence of much more complex differential equations is obvious, are out of the scope of the present work. It is worth to mention that, by reading back the results for the susy partners of the harmonic and radial oscillators, explicit solutions of these Painlevé equations can be immediately supplied, a simple fact which is not well known in the mathematical literature.

The paper is organized as follows. In section 2 we discuss the polynomial deformations of the Heisenberg algebra, in particular, the possible spectra which can be found. In section 3 the higher order supersymmetric quantum mechanics (hsusy QM) will be introduced, and the corresponding susy partners of the harmonic and radial oscillators will be analyzed. We will look for the general systems ruled by the polynomial Heisenberg algebras in section 4, where we will realize the growing complexity arising as \( m \) is increased. We finish the paper with some conclusions and an outlook of future work.

2. Polynomial deformations of the Heisenberg algebra

Polynomial Heisenberg algebras of \( m \)-th order are deformations of the oscillator algebra, where there are two standard commutation relationships

\[
[H, L^\pm] = \pm L^\pm, \tag{2.1}
\]

and an atypical one characterizing the deformation:

\[
[L^-, L^+] \equiv N(H + 1) - N(H) = P_m(H), \tag{2.2}
\]

where the generalized number operator is \( N(H) \equiv L^+ L^- \). The corresponding systems are described by the Schrödinger Hamiltonian

\[
H = -\frac{1}{2} \frac{d^2}{dx^2} + V(x), \tag{2.3}
\]

where \( L^\pm \) are \((m + 1)\)-th order differential ladder operators, \( N(H) \) is a polynomial in \( H \) factorized as

\[
N(H) = \prod_{i=1}^{m+1} (H - E_i), \tag{2.4}
\]

and \( P_m(H) \) in (2.2) is a \( m \)-th order polynomial in \( H \). The algebraic structure generated by \( \{H, L^-, L^+\} \) provides information on the spectrum \( \text{Sp}(H) \) of \( H \) [6, 11, 14, 12]. Indeed, let us consider the solution space of the \((m + 1)\)-th order differential equation (the kernel \( K_{L^-} \) of \( L^- \)):

\[
L^- \psi = 0 \quad \Rightarrow \quad L^+ L^- \psi = \prod_{i=1}^{m+1} (H - E_i) \psi = 0. \tag{2.5}
\]

As \( K_{L^-} \) is invariant under \( H \), it is natural to select as the basis of \( K_{L^-} \) these solutions which are simultaneously eigenstates of \( H \) with eigenvalues \( E_i \)

\[
H \psi_{E_i} = E_i \psi_{E_i}, \tag{2.6}
\]

becoming the extremal states of the \( m + 1 \) mathematical ladders of spacing \( \Delta E \equiv 1 \) starting from \( E_i \). If \( s \) of these states are physically meaningful, \( \{\psi_{E_i}, i = 1, \ldots, s\} \), then by acting iteratively with \( L^+ \), \( s \) physical energy ladders can be constructed (see Figure 1 (a)).
3. Higher order supersymmetric quantum mechanics

Let two Schrödinger Hamiltonians $H_0, H_k$ of the form (2.3) be intertwined by differential operators $B^\dagger, B$ of $k$-th order [15, 16, 17, 18, 11, 12, 19]

$$H_k B^\dagger = B^\dagger H_0, \quad H_0 B = BH_k,$$

(3.1)

where $B^\dagger$ is the adjoint of $B$ and the Hamiltonians are assumed to be self-adjoint. The standard supersymmetry algebra

$$[Q_i, H_{ss}] = 0, \quad \{Q_i, Q_j\} = \delta_{ij} H_{ss}, \quad i, j = 1, 2,$$

(3.2)

is realized by choosing

$$Q = \begin{pmatrix} 0 & B^\dagger \\ 0 & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix},$$

(3.3)

$$Q_1 = \frac{Q^\dagger + Q}{\sqrt{2}}, \quad Q_2 = \frac{Q^\dagger - Q}{\sqrt{2}}, \quad H_{ss} = \begin{pmatrix} B\dagger B & 0 \\ 0 & BB^\dagger \end{pmatrix}.$$  

(3.4)
In this so called \( k \)-th order supersymmetric quantum mechanics \((k\text{-susy QM})\) there is a polynomial relationship between \( H_{ss} \) and the diagonal matrix \( H_d \) involving \( H_0 \) and \( H_k \):

\[
H_d = \begin{pmatrix} H_k & 0 \\ 0 & H_0 \end{pmatrix}, \quad H_{ss} = (H_0 - \epsilon_1) \cdots (H_0 - \epsilon_k).
\] (3.5)

The standard susy QM is obtained through the first order intertwining operator

\[
B^I = A_1^I = \frac{1}{\sqrt{2}} \left[ \frac{d}{dx} + \alpha_1(x, \epsilon_1) \right],
\] (3.6)

which leads to the typical relation between the potentials \( V_0(x) \) and \( V_1(x) \):

\[
V_1(x) = V_0(x) - \alpha_1^2(x, \epsilon_1),
\] (3.7)

where \( \alpha_1(x, \epsilon_1) \) satisfies the Riccati equation:

\[
\alpha_1(x, \epsilon_1) + \alpha_1^2(x, \epsilon_1) = 2[V_0(x) - \epsilon_1].
\] (3.8)

For a given potential \( V_0(x) \) and factorization energy \( \epsilon_1 \), the generation of \( V_1(x) \) requires either to solve (3.8) or the corresponding Schrödinger equation

\[
-\frac{d^2}{dx^2} + V_0(x)u_1 = \epsilon_1 u_1, \quad \alpha_1(x, \epsilon_1) = \frac{u_1'}{u_1},
\] (3.9)

On the other hand, if \( B^I \) is of order \( k > 1 \) the potential \( V_k(x) \) can be found either through Crum determinants [16] or by defining a sequence of Hamiltonians \( H_0, \ldots, H_k \) intertwined by first order operators \( A_i^I = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dx} + \alpha_i(x, \epsilon_i) \right] \) [17, 18]:

\[
H_i A_i^I = A_i^I H_{i-1}, \quad i = 1, \ldots, k.
\] (3.10)

Taking into account (3.6)–(3.7), the \( i \)-th potential reads \( V_i(x) = V_{i-1}(x) - \alpha_i^2(x, \epsilon_i) \), where:

\[
\alpha_i^2(x, \epsilon_i) + \alpha_i^2(x, \epsilon_i) = 2[V_{i-1}(x) - \epsilon_i].
\] (3.11)

By adopting the identifications

\[
B^I = A_k^I \cdots A_1^I, \quad B = A_1 \cdots A_k,
\] (3.12)

it turns out that the final potential reads:

\[
V_k(x) = V_0(x) - \sum_{i=1}^k \alpha_i^2(x, \epsilon_i).
\] (3.13)

The corresponding \( \alpha_i \)'s are found through the Bäcklund type transformation [7, 8, 17]:

\[
\alpha_i(x, \epsilon_i) = -\alpha_{i-1}(x, \epsilon_{i-1}) - \frac{2(\epsilon_{i-1} - \epsilon_i)}{\alpha_{i-1}(x, \epsilon_{i-1}) - \alpha_{i-1}(x, \epsilon_i)}.
\] (3.14)

Its iterations show that the right hand side of (3.13) depends just of \( k \) solutions \( \alpha_1(x, \epsilon_i) \) of the first Riccati equation (3.11) with factorization energies \( \epsilon_i, i = 1, \ldots, k \).

The hsusy QM is useful to generate solvable potentials from a given initial one. Moreover, we will see next that the hsusy partners of the harmonic and radial oscillators realize in a natural way the polynomial Heisenberg algebras.
3.1. Higher order susy partners of the harmonic oscillator

Let us consider the harmonic oscillator potential $V_0(x) = x^2/2$. The corresponding energy eigenvalues and eigenfunctions are $E_{n}^{(0)} = n + \frac{1}{2}$, $\psi_{n}^{(0)}(x) \propto e^{-x^2/2} H_n(x)$, $n = 0, 1, \ldots$, where $H_n(x)$ are Hermite polynomials. The standard ladder operators $a = \frac{1}{\sqrt{2}}(\frac{d}{dx} + x)$, $a^\dagger = \frac{1}{\sqrt{2}}(-\frac{d}{dx} + x)$ connect the $\psi_{n}^{(0)}$'s as follows: $a\psi_{n}^{(0)} = \sqrt{n}\psi_{n-1}^{(0)}$, $a^\dagger\psi_{n}^{(0)} = \sqrt{n+1}\psi_{n+1}^{(0)}$.

The algebra generated by $\{H_0, a, a^\dagger\}$ is of the type (2.1)--(2.4), with $m = 0$ and $E_1 = \frac{1}{2}$.

In order to generate the susy partner potentials $V_k(x)$ by creating $k$ new levels $\epsilon_1, \ldots, \epsilon_k$ below $E_0^{(0)}$, we need the general solution to the Schrödinger equation (3.9). Up to a constant factor, we have [11]:

$$u_1(x) = e^{-\frac{x^2}{2}} \left[ _1F_1 \left( \frac{1-2\epsilon_1}{4}, \frac{1}{2}; x^2 \right) + 2x \nu_1 \frac{\Gamma(\frac{3-2\epsilon_1}{4})}{\Gamma(\frac{1-2\epsilon_1}{4})} _1F_1 \left( \frac{3-2\epsilon_1}{4}, \frac{3}{2}; x^2 \right) \right].$$

(3.15)

To avoid singularities for the 1-susy case we must have $|\nu_1| < 1$. The corresponding $\nu_1$-restriction in the higher order situation is, in general, different. The eigenfunctions of $H_k$ are found by applying the $B^\dagger$ of (3.12) to the oscillator eigenfunctions. Thus, the spectrum is $\text{Sp}(H_k) = \{\epsilon_i, E_{\epsilon_i}^{(k)} \}, i = 1, \ldots, k, n = 0, 1, \ldots$, a fact that can be explained by means of the polynomial algebra (2.1)--(2.4). Indeed, the natural ladder operators for $H \equiv H_k$ are [20, 11]:

$$L^- = B^\dagger a B, \quad L^+ = B^\dagger a^\dagger B,$$

(3.16)

where $B, B^\dagger$ are the intertwining operators of (3.12). As $L^-$ and $L^+$ are of $(2k+1)$-th order, it turns out that $\mathcal{N}(H) = L^+ L^-$ is a $(2k+1)$-th order polynomial in $H$, namely:

$$\mathcal{N}(H) = \left( H - \frac{1}{2} \right) \prod_{i=1}^{k} (H - \epsilon_i - 1)(H - \epsilon_i).$$

(3.17)

The pairs of roots $\{\epsilon_i, \epsilon_i + 1\}$ in (3.17) imply the existence of $k$ one-step ladders in $\text{Sp}(H)$, one at each $\epsilon_i$. There is also an infinite one starting from $\frac{1}{2}$. Since $[L^-, L^+] = P_{2k}(H)$, we see that the even order polynomial algebras (2.1)--(2.4) are realized naturally by the susy partners of the harmonic oscillator [11].

3.2. Higher order susy partners of the radial oscillator

Let us consider now the potential $V_0(x) = \frac{x^2}{8} + \frac{l(l+1)}{2x^2}$, $x > 0, \quad l \geq 0$.

(3.18)

Throughout this paper we will refer to this system as the radial oscillator. It is known that its spectrum can be built up algebraically using the following second order ladder operators $A^- A^+$ [21, 19]:

$$A^- = \frac{1}{2} \left( \frac{d^2}{dx^2} + x \frac{d}{dx} + \frac{x^2}{4} - \frac{l(l+1)}{x^2} + \frac{1}{2} \right),$$

(3.19)

$$A^+ = \frac{1}{2} \left( \frac{d^2}{dx^2} - x \frac{d}{dx} + \frac{x^2}{4} - \frac{l(l+1)}{x^2} - \frac{1}{2} \right).$$

(3.20)

Two ladders can be constructed out of the two eigenstates of $H_0$ annihilated by $A^-$. The physical ladder starts from the extremal state

$$\psi_{E_1}^{(0)}(x) \propto x^{l+1} e^{-\frac{x^2}{2}},$$

(3.21)
which is square-integrable in $[0, \infty)$ and vanishes at the extremes of that interval. (The last is the standard requirement for systems with a singularity at $x = 0$ of kind (3.18); we will adopt here this boundary condition in order to characterise the spectrum of $H_0$.) The associated eigenvalue is $E_1 = \frac{3}{2} + \frac{\lambda}{4} \equiv E_0^{(0)}$, and the next eigenstates are obtained by acting with powers of $A^+$ on $\psi_0^{(0)} \equiv \psi_0^{(0)}$. The second ladder departs from the other extremal state

$$\psi_{E_2}^{(0)}(x) \propto x^{\frac{l}{2} - \frac{\lambda^2}{8}}$$

(3.22)

of eigenvalue $E_2 = -\frac{l}{2} + \frac{\lambda}{4}$. This state is unphysical because, at $x = 0$, $\psi_{E_2}^{(0)}(x)$ diverges for $l > 0$ and it does not vanish for $l = 0$. Thus, $\text{Sp}(H_0) = \{E_n^{(0)} = n + \frac{l}{2} + \frac{\lambda}{4}, n = 0, 1, \ldots\}$.

To implement now the susy techniques, we solve the Schrödinger equation (3.9) with the potential (3.18). Up to a constant factor, the general solution is given by [19]:

$$u_1(x) = \frac{\Gamma(\frac{3}{2} - l)\Gamma(\frac{l}{2} - 1)}{\Gamma(\frac{3}{2})}\left[1_{1F1}\left(\frac{3 + 2l - 4\epsilon_1}{4}, \frac{3 + 2l - 4\epsilon_1}{2}, \frac{\lambda^2}{2}\right) + \nu_1 \left(1_{1F1}\left(\frac{1}{4}, \frac{3 + 2l - 4\epsilon_1}{4}, \frac{3 + 2l - 4\epsilon_1}{2}\right)\right)\right].$$

(3.23)

To avoid 1-susy singularities in the domain $x > 0$, we must take $\epsilon_1 \leq E_0^{(0)}$ and $\nu_1 \geq -\Gamma(\frac{1}{2} - l)/\Gamma(\frac{1}{2} - \frac{l}{2} - \epsilon_1)$. This restriction on $\nu_1$ changes in the higher order case.

If the susy QM is used to create $k$ new levels $\epsilon_i \leq E_0^{(0)}$, we will have $\text{Sp}(H_k) = \{\epsilon_i, E_n^{(0)} = n + \frac{l}{2} + \frac{\lambda}{4}, i = 1, \ldots, k, n = 0, 1, \ldots\}$, i.e., the polynomial algebra (2.1)–(2.4) rules the hussy partners of the radial oscillator, with natural ladder operators given by

$$L^- = B^\dagger A^- B, \quad L^+ = B^\dagger A^+ B.$$  

(3.24)

As $A^\pm$ are second order operators and $B, B^\dagger$ are $k$-th order ones, then $L^-$ and $L^+$ are of order $(2k + 2)$ implying that $N(H) = L^+L^-$ is a $(2k + 2)$-th order polynomial in $H$:

$$N(H) = \left(H - \frac{l}{2} - \frac{3}{4}\right)\left(H + \frac{l}{2} - \frac{1}{4}\right)\prod_{i=1}^{k}(H - \epsilon_i)(H - \epsilon_i - 1).$$

(3.25)

The pair of roots $\{\epsilon_i, \epsilon_i+1\}$ indicate the existence of $k$ physical one-step ladders situated at $\epsilon_i$. There is also one infinite physical ladder with lower end $E_0^{(0)}$. Since $[L^-, L^+] = 2e_{k+1}(H)$, it is seen that the hussy partners of the radial oscillator realize naturally the polynomial algebras (2.1)–(2.4) of odd order.

Up to now we have constructed systems ruled by the polynomial algebra (2.1)–(2.4) through the hussy QM. Now, we will look for the most general systems described by such deformed algebras.

4. Polynomial Heisenberg algebras: general systems

Let us determine the general systems described by the polynomial algebras (2.1)–(2.4). Since for $m$ greater than 4 the calculations are quite involved, we will analyse just the cases with $m = 0, 1, 2, 3$. 

\[\text{\textit{Polynomial Heisenberg algebras}}\]
4.1. Ladder operators of first order \((m = 0)\)

We look for the general Hamiltonian (2.3) and first order ladder operators

\[
L^+ = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dx} + f(x) \right], \quad L^- = (L^+)^\dagger, \quad (4.1)
\]

satisfying equation (2.1). Thus, a system involving \(V, f,\) and their derivatives is obtained:

\[
f' - 1 = 0, \quad V' - f = 0. \quad (4.2)
\]

Up to coordinate and energy displacements, it turns out that \(f(x) = x\) and \(V(x) = x^2/2\). This potential has one equidistant infinite ladder starting from the extremal state \(\psi_\mathcal{E} = \pi^{-1/4} e^{-x^2/2}\), which is annihilated by \(L^-\) and it is a normalized eigenfunction of \(H\) with eigenvalue \(\mathcal{E} = \frac{1}{2}\).

Here, the number operator is linear in \(H\), \(N(H) = H - \mathcal{E}\), i.e., the general system obeying the polynomial Heisenberg algebra (2.1)-(2.4) with \(m = 0\) is the harmonic oscillator.

4.2. Second order ladder operators \((m = 1)\)

Let us suppose now that

\[
L^+ = \frac{1}{2} \left[ \frac{d^2}{dx^2} + g(x) \frac{d}{dx} + h(x) \right], \quad L^- = (L^+)^\dagger. \quad (4.3)
\]

Equation (2.1) leads then to a system of equations for \(V, g, h,\) and their derivatives:

\[
g' + 1 = 0, \quad h' + 2V' + g = 0,
\]

\[
h'' + 2V'' + 2gV' + 2h = 0. \quad (4.4)
\]

The general solution (up to coordinate and energy displacements) is given by:

\[
g(x) = -x, \quad h(x) = \frac{x^2}{4} - \frac{k}{x^2} - \frac{1}{2}, \quad V(x) = \frac{x^2}{8} + \frac{k}{2x^2}. \quad (4.4)
\]

The potentials (4.4) have two equidistant energy ladders (not necessarily physical) generated by acting the powers of \(L^+\) onto the two extremal states:

\[
\psi_{\mathcal{E}_1} \propto x^{1+\sqrt{k+\frac{1}{4}}} e^{-\frac{x^2}{2}}, \quad \psi_{\mathcal{E}_2} \propto x^{1-\sqrt{k+\frac{1}{4}}} e^{-\frac{x^2}{2}}. \quad (4.5)
\]

Let us remind that \(L^-\psi_{\mathcal{E}_i} = 0 = (H - \mathcal{E}_i)\psi_{\mathcal{E}_i},\) where \(\mathcal{E}_1 = \frac{1}{2} + \frac{1}{2}\sqrt{k + \frac{1}{4}}, \quad \mathcal{E}_2 = \frac{1}{2} - \frac{1}{2}\sqrt{k + \frac{1}{4}}.\) Now \(N(H)\) is quadratic in \(H: N(H) = (H - \mathcal{E}_1)(H - \mathcal{E}_2).\) The potentials (3.18) are recovered by making \(k = l(l + 1), l \geq 0.\) Thus, the general systems having second order ladder operators are described by the radial oscillator potentials (4.4).

4.3. Ladder operators of third order \((m = 2)\)

Let \(L^\pm\) be now third order ladder operators, factorized by convenience as [12]:

\[
L^+ = L^+_1 L^+_2, \quad L^- = L^-_2 L^-_1, \quad (4.6)
\]

where \(L^-_1 = (L^+_1)^\dagger, L^-_2 = (L^+_2)^\dagger\) and

\[
L^+_1 = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dx} + f(x) \right], \quad L^+_2 = \frac{1}{2} \left[ \frac{d^2}{dx^2} + g(x) \frac{d}{dx} + h(x) \right]. \quad (4.7)
\]

It is assumed the existence of an auxiliary Hamiltonian \(H_a\) intertwined with \(H\) as follows

\[
HL^+_1 = L^+_1 (H_a + 1), \quad H_a L^-_2 = L^-_2 H. \quad (4.8)
\]
Thus, we arrive to the following system of equations:

\[ -f' + f^2 = 2V - 2\mathcal{E}_3, \quad (4.9) \]
\[ V_1 = V + f' - 1 = V + g', \quad (4.10) \]
\[ \frac{g''}{2g} = \left( \frac{g'}{2g} \right)^2 - g' + \frac{g^2}{4} + \frac{(\mathcal{E}_1 - \mathcal{E}_2)^2}{g^2} + \mathcal{E}_1 + \mathcal{E}_2 - 2 = 2V. \quad (4.11) \]

By integrating (4.10) (up to a coordinate displacement) we get

\[ f(x) = g(x) + x, \quad (4.12) \]
and using (4.11)–(4.12) in (4.9), we find the following differential equation for \( g(x) \)

\[ g'' = \frac{g^2}{2g} + \frac{3}{2}g^3 + 4xg^2 + 2(x^2 + 2\mathcal{E} + 1) g - \frac{2\Delta_1^2}{g}, \quad (4.13) \]

which is a Painlevé IV (PIV) equation with \( \Delta_1 = \mathcal{E}_1 - \mathcal{E}_2, \mathcal{E} = \mathcal{E}_3 - \frac{1}{2}(\mathcal{E}_1 + \mathcal{E}_2) \) [22, 7, 8, 23, 12, 13]. The potential \( V(x) \) can be found by substituting (4.12) into (4.9):

\[ V(x) = \frac{x^2}{2} - \frac{g'}{2} + \frac{g^2}{2} + xg + \mathcal{E}_3 - \frac{1}{2}, \quad (4.14) \]

It has three energy ladders, each one with equidistant levels. The extremal states are such that \( L^- \psi_{\mathcal{E}_i} = (H - \mathcal{E}_i) \psi_{\mathcal{E}_i} = 0, i = 1, 2, 3, \) and are given by:

\[ \psi_{\mathcal{E}_1} \propto \left( \frac{g'}{2g} - \frac{g}{2} - \frac{\Delta_1}{g} - x \right) \exp \left[ \int \left( \frac{g'}{2g} + \frac{g}{2} - \frac{\Delta_1}{g} \right) dx \right], \quad (4.15a) \]
\[ \psi_{\mathcal{E}_2} \propto \left( \frac{g'}{2g} - \frac{g}{2} + \frac{\Delta_1}{g} - x \right) \exp \left[ \int \left( \frac{g'}{2g} + \frac{g}{2} + \frac{\Delta_1}{g} \right) dx \right], \quad (4.15b) \]
\[ \psi_{\mathcal{E}_3} \propto \exp \left( -\frac{x^2}{2} - \int gdx \right). \quad (4.15c) \]

Here the generalized number operator is cubic in \( H \):

\[ N(H) = (H - \mathcal{E}_1)(H - \mathcal{E}_2)(H - \mathcal{E}_3). \quad (4.16) \]

Hence, we have a recipe for building systems ruled by second order polynomial algebras (2.1)–(2.4): first find a \( g(x) \) obeying the PIV equation (4.13); then calculate the potential, using (4.14), and its three ladders from the extremal states (4.17). In order to test the effectivity of this recipe, let us analyse some systems associated to particular PIV solutions \( g(x) \).

\textit{i) The harmonic oscillator.} Let us consider the following solution of (4.13):

\[ g(x) = -x - \alpha(x), \quad (4.17) \]

where \( \mathcal{E}_1 = \mathcal{E}_3, \alpha(x) = u'/u \) satisfies the Riccati equation

\[ \alpha'(x) + \alpha^2(x) = x^2 - 2\mathcal{E}, \quad \epsilon = 2\mathcal{E} + \frac{1}{2} = \mathcal{E}_3 - \mathcal{E}_2 + \frac{1}{2}, \quad (4.18) \]

and \( u(x) \) is the Schrödinger solution given in (3.15). This \( g(x) \) substituted in (4.14) provides:

\[ V(x) = \frac{x^2}{2} + \mathcal{E}_2 - \frac{1}{2}, \quad (4.19) \]

which is the harmonic oscillator potential. The three extremal states (4.15a)–(4.15c) become:

\[ \psi_{\mathcal{E}_1} = 0, \quad \psi_{\mathcal{E}_2} \propto e^{-\frac{x^2}{2}}, \quad \psi_{\mathcal{E}_3} \propto u(x). \quad (4.20) \]

We see that the only physical ladder is the one generated from \( \psi_{\mathcal{E}_2} \). Here we have a case where the deformed algebra is reducible in the sense explained at the end of section 2. In fact it is easy to show that actually \( L^- = a^- (H - \mathcal{E}_1) \).
ii) The 1-susy oscillator partners. They arise for \( g(x) \) taking the form:

\[
g(x) = -x + \alpha(x),
\]

where \( \alpha = u' / u \) satisfies (4.18), but now \( \mathcal{E}_1 = \mathcal{E}_3 + 1, \epsilon = 2\epsilon + \frac{3}{2} = \mathcal{E}_3 - \mathcal{E}_2 + \frac{1}{2} \), and \( u(x) \) is again the Schrödinger solution (3.15). This \( g(x) \) leads to the exactly solvable potentials:

\[
V(x) = \frac{x^2}{2} - \alpha'(x) + \mathcal{E}_2 - \frac{1}{2},
\]

which are the 1-susy partners of the oscillator. The extremal states become:

\[
\psi_{\mathcal{E}_1} \propto B^\dagger \alpha^\dagger u(x), \quad \psi_{\mathcal{E}_2} \propto B^\dagger e^{-\frac{x^2}{2}}, \quad \psi_{\mathcal{E}_3} \propto \frac{1}{u(x)},
\]

where \( B^\dagger \) is a first order intertwiner, as in (3.6).

iii) The \( k \)-susy oscillator partners with \( k > 1 \). Recently, it has been found a method which allows to connect the \( k \) independent one-step ladders of the \( k \)-susy Hamiltonians \( \mathcal{H} \equiv \mathcal{H}_k \) of section 3.1, to build just a ladder with \( k \) steps [24]. The corresponding systems, in principle ruled by the \( (2k) \)-th order deformed structures (2.1)–(2.4), will be described now by a polynomial Heisenberg algebra of second order, supplying us with more solutions to the PIV equation. The process consists in taking \( k \) transformation functions \( u_i \) of an unphysical ladder, i.e.,

\[
H \partial u_i = \epsilon_i u_i, \quad u_{i+1} \propto \alpha^\dagger u_i \epsilon_i = \epsilon_1 + i - 1 < \frac{1}{2}, \quad i = 1, \ldots, k,
\]

with \( u_1 \) being the Schrödinger solution in (3.15). With this choice, the \( k-1 \) factorization energies \( \epsilon_2 = \epsilon_1 + 1, \ldots, \epsilon_k = \epsilon_1 + k - 1 \) appear twice in (3.17), implying that the natural \( (2k + 1) \)-th order ladder operator of section 3.1 can be written as the product of \( (\mathcal{H} - \epsilon_2) \cdots (\mathcal{H} - \epsilon_k) \) times a third order operator [24]. This operator leads to the PIV solution we are looking for. Moreover, from the extremal states in the two previous cases it is clear that now

\[
\psi_{\mathcal{E}_1} \propto B^\dagger \alpha^\dagger u_k(x), \quad \psi_{\mathcal{E}_2} \propto B^\dagger e^{-\frac{x^2}{2}}, \quad \psi_{\mathcal{E}_3} \propto \frac{W(u_2, \ldots, u_k)}{W(u_1, \ldots, u_k)},
\]

where \( B^\dagger \) is the \( k \)-th order intertwining operator of (3.12), \( \alpha = u_k' / u_1 \) satisfies (4.18) but with \( \mathcal{E}_1 = \mathcal{E}_3 + k, \epsilon = 2\epsilon + \frac{3}{2} + k = \mathcal{E}_3 - \mathcal{E}_2 + \frac{1}{2} \), and the expression for \( \psi_{\mathcal{E}_3} \) will be justified in the Appendix. By comparing (4.15c) with (4.25) it turns out that the solution \( g(x) \) of the PIV equation (4.13) reads now

\[
g(x) = -x - [\ln \psi_{\mathcal{E}_3}]' = -x - [\ln W(u_2, \ldots, u_k)]' + [\ln W(u_1, \ldots, u_k)]'
\]

and the corresponding potential becomes (see an example in Figure 3)

\[
V(x) = \frac{x^2}{2} - [\ln W(u_1, \ldots, u_k)]'' + \mathcal{E}_2 - \frac{1}{2}.
\]

4.4. Fourth order ladder operators \((m = 3)\)

Let \( L^\pm \) be fourth order ladder operators factorized as in (4.6) and obeying (4.8) but now:

\[
L^+_1 = \frac{1}{2} \left[ \frac{d^2}{dx^2} + g_1(x) \frac{d}{dx} + h_1(x) \right].
\]
An explicit calculation leads to the system of equations

$$\frac{g'_{1}}{2g_{1}} = \left( \frac{g'_{1}}{2g_{1}} \right)^{2} + \frac{g'_{1}}{4} + \frac{(\mathcal{E}_{3} - \mathcal{E}_{4})^{2}}{g_{1}^{2}} + \mathcal{E}_{3} + \mathcal{E}_{4} = 2V;$$  \hspace{1cm} (4.29)

$$\mathcal{V}_{4} = V - g'_{1} - 1 = V + g', \hspace{1cm} (4.30)$$

$$\frac{g'_{0}}{2g} = \left( \frac{g'_{0}}{2g} \right)^{2} - g' + \frac{g^{2}}{4} + \frac{(\mathcal{E}_{1} - \mathcal{E}_{2})^{2}}{g^{2}} + \mathcal{E}_{1} + \mathcal{E}_{2} - 2 = 2V; \hspace{1cm} (4.31)$$

From (4.30) we get, up to a coordinate displacement,

$$g_{1}(x) = -g(x) - x. \hspace{1cm} (4.32)$$

By substituting (4.31)–(4.32) into (4.29) one arrives to the differential equation for $g(x)$:

$$g'' = \frac{2g + x}{2g(g + x)}(g')^{2} - \frac{g}{x(g + x)}g' + R(x, g), \hspace{1cm} (4.33)$$
with
\[ R(x,g) = (2xg(g + x))^{-1}[2xg^5 + (5x^2 + 8\mathcal{E} + 4)g^4 + 4x(x^2 + 4\mathcal{E} + 2)g^3 \]
\[ + [x^4 + 4(2\mathcal{E} + 1)x^2 + 4(\Delta_2^2 - \Delta_1^2) - 1]g^2 - 4\Delta_1^2 x(2g + x)], \]  
where \( \mathcal{E} = \frac{1}{4}(\mathcal{E}_3 + \mathcal{E}_4) - \frac{1}{2}(\mathcal{E}_1 + \mathcal{E}_2), \Delta_1 = \mathcal{E}_1 - \mathcal{E}_2 \) and \( \Delta_2 = \mathcal{E}_3 - \mathcal{E}_4. \) In order to identify equation (4.33), let us make \( g = x/(w - 1) \) and then change the variable as \( z = x^2. \) Thus,
\[ \frac{d^2 w}{dz^2} = \left( \frac{1}{2w} + \frac{1}{w - 1} \right) \left( \frac{dw}{dz} \right)^2 - \frac{1}{z} \frac{dw}{dz} + \frac{(w - 1)^2}{z^2} \left( \frac{cw^2 + b}{w} \right) + \frac{cw}{z} + \frac{dw(w + 1)}{w - 1}, \]  
which is a Painlevé V equation (PV) with \( a = \frac{\Delta_2^2}{2}, b = -\frac{\Delta_2^2}{2}, c = -\mathcal{E} - \frac{1}{2}, d = -\frac{1}{8} \) [22, 8]. The spectrum contains four independent equidistant ladders starting from the extremal states
\[ \psi_{\varepsilon_1} \propto \left[ \frac{g_1}{2g} \left( \frac{g'}{g} - \frac{g_1'}{g_1} + \frac{g + g_1}{2} - \frac{\Delta_1}{g} \right) + \mathcal{E} - \frac{\Delta_1}{2} \right] \exp \left[ \int \left( \frac{g'}{2g} + \frac{g + \Delta_1}{2} \right) dx \right], \]
\[ \psi_{\varepsilon_2} \propto \left[ \frac{g_1}{2g} \left( \frac{g'}{g} - \frac{g_1'}{g_1} + \frac{g + g_1}{2} + \frac{\Delta_1}{g} \right) + \mathcal{E} + \frac{\Delta_1}{2} \right] \exp \left[ \int \left( \frac{g'}{2g} + \frac{g - \Delta_1}{2} \right) dx \right], \]
\[ \psi_{\varepsilon_3} \propto \exp \left[ \int \left( \frac{g_1}{2g_1} + \frac{g_1}{2} - \frac{\Delta_2}{g_1} \right) dx \right], \]
\[ \psi_{\varepsilon_4} \propto \exp \left[ \int \left( \frac{g_1}{2g_1} + \frac{g_1}{2} + \frac{\Delta_2}{g_1} \right) dx \right]. \]  
The number operator is a fourth order polynomial in \( H: \)
\[ N(H) = (H - \varepsilon_1)(H - \varepsilon_2)(H - \varepsilon_3)(H - \varepsilon_4). \]  
Hence, given a solution \( u(z) \) of the PV equation (4.35) (or \( g(x) \) for (4.33)), a system characterised by the third order polynomial algebra (2.1)–(2.4) with potential (4.31) and extremal states (4.36) is straightforwardly constructed. In order to test our recipe, let us discuss some explicit examples.

\textit{a) The radial oscillator.} Let us consider the following solution of equation (4.33)
\[ g(x) = -\frac{x}{2} - \frac{I}{x} - \alpha(x), \]  
where \( \varepsilon_1 = \mathcal{E}_1, l = 2\mathcal{E} + \frac{1}{2} = \mathcal{E}_4 - \mathcal{E}_2 + \frac{1}{2} \) and \( \alpha(x) \) is a solution of the Riccati equation
\[ \alpha(x) + \alpha^2(x) = 2 \left[ \frac{x^2}{8} + \frac{l(l + 1)}{2x^2} - \epsilon \right], \]  
being \( \epsilon = (\Delta_2^2 - \Delta_1^2)/(4\mathcal{E}) = \mathcal{E}_3 - (\mathcal{E}_2 + \mathcal{E}_4)/2. \) The function \( \alpha(x) \) can be written as \( \alpha(x) = x' / u, \) with \( u \) given by (3.23). The corresponding potentials (4.31) become:
\[ V(x) = \frac{x^2}{8} + \frac{l(l - 1)}{2x^2} + \mathcal{E}_4 + \mathcal{E}_3 - \frac{1}{2}, \]  
i.e., the radial oscillator. Its corresponding extremal states (4.36) read:
\[ \psi_{\varepsilon_1} = 0, \quad \psi_{\varepsilon_2} \propto x^{-(l-1)}e^{-\frac{x^2}{4}}, \quad \psi_{\varepsilon_3} \propto \left[ u' - \left( \frac{x}{2} - \frac{l}{x} \right) u \right], \quad \psi_{\varepsilon_4} \propto x^l e^{-\frac{x^2}{4}}. \]  
The fact that the radial oscillator is a system described by a Lie algebra and by a deformed algebra at the same time tell us that this deformed algebra must be reducible in the same way as mentioned in example i).
b) The 1-susy partners of the radial oscillator. If we take now
\[ g(x) = -\frac{x}{2} - \frac{(l+1)}{x} + \alpha(x), \]  
(4.42)
it turns out that \( \alpha = u' / u \) satisfies again the Riccati equation (4.39), but now \( \mathcal{E}_1 = \mathcal{E}_3 + 1 \), \( l = 2\mathcal{E} + \frac{1}{2} = \mathcal{E}_4 - \mathcal{E}_2 - \frac{1}{2} \), \( \epsilon = (\Delta_1^2 - \Delta_0^2)/(4(\mathcal{E} + 1)) = \mathcal{E}_3 - (\mathcal{E}_2 + \mathcal{E}_4 - 1)/2 \), and \( u \) given in (3.23). The potentials (4.31) become now:
\[ V(x) = \frac{x^2}{8} + \frac{l(l+1)}{2x^2} + \frac{\mathcal{E}_2 + \mathcal{E}_4 - 1}{2} - \alpha'(x), \]  
(4.43)
i.e., the 1-susy partners of the radial oscillator. The four extremal states read:
\[ \psi_{\mathcal{E}_1} \propto B^\dagger A^\dagger u_1, \quad \psi_{\mathcal{E}_2} \propto B^\dagger \left( x^{-l}e^{-\frac{x^2}{4}} \right), \]  
(4.44a)
\[ \psi_{\mathcal{E}_3} \propto \frac{1}{u}, \quad \psi_{\mathcal{E}_4} \propto B^\dagger \left( x^{l+1}e^{-\frac{x^2}{4}} \right), \]  
(4.44b)
where \( B^\dagger \) is the first order intertwiner and \( A^\dagger \) is the second order ladder operator in (3.20).

c) The k-susy radial oscillator partners with \( k > 1 \). As in the example iii) of section 4.3, a reduction process allows to assemble the one-step ladders of certain \( k \)-susy partner Hamiltonians of the radial oscillator. Then, the natural \( (2k+1) \)-th order polynomial algebra ruling \( H \equiv H_k \) reduces to a third order one, leading then to solutions of the PV equation. Indeed, let us take the \( k \) transformation functions once again as the steps of an unphysical ladder of the radial oscillator potential \( V_0(x) \) in (3.18), i.e.,
\[ H_0 u_i = \epsilon_i u_i, \quad u_{i+1} \propto A^\dagger u_i, \quad \epsilon_i = i\epsilon_1 + i - 1 < \frac{l}{2} + \frac{3}{4}, \quad i = 1, \ldots, k. \]  
(4.45)
From the previous example b), one immediately finds the four extremal states associated to the reduced third order polynomial algebra:
\[ \psi_{\mathcal{E}_1} \propto B^\dagger A^\dagger u_k, \quad \psi_{\mathcal{E}_2} \propto B^\dagger \left( x^{-l}e^{-\frac{x^2}{4}} \right), \quad \psi_{\mathcal{E}_3} \propto \frac{W(u_1, \ldots, u_k)}{W(u_1, \ldots, u_k)}, \]  
(4.46a)
\[ \psi_{\mathcal{E}_4} \propto B^\dagger \left( x^{l+1}e^{-\frac{x^2}{4}} \right) \propto \frac{W(u_1, \ldots, u_k, x^{l+1}e^{-\frac{x^2}{4}})}{W(u_1, \ldots, u_k)}, \]  
(4.46b)
where now \( B^\dagger \) is the \( k \)-th order intertwiner operator of (3.12), \( \alpha = u'_1 / u_1 \) satisfies again (4.39) but with \( \epsilon = (\Delta_1^2 - \Delta_0^2)/(4(\mathcal{E} + k)) = (k - 1)/2 = \mathcal{E}_3 - (\mathcal{E}_2 + \mathcal{E}_4 - 1)/2 \), \( \mathcal{E}_1 = \mathcal{E}_3 + k, l = 2\mathcal{E} - 1/2 + k = \mathcal{E}_4 - \mathcal{E}_2 - \frac{1}{2} \), and \( u_1 \) is the solution (3.23). The last formula in (4.46a) will be discussed in the Appendix. By comparing the last of the four equations in (4.36) with (4.46b) we immediately obtain:
\[ g_1(x) = \frac{2\Delta_2}{\ln \left( \frac{\psi_{\mathcal{E}_4}}{\psi_{\mathcal{E}_3}} \right)} = \frac{2\Delta_2 W(u_2, \ldots, u_k) W(u_1, \ldots, u_k, x^{l+1}e^{-\frac{x^2}{4}})}{W(u_1, \ldots, u_k)}, \]  
(4.47)
Therefore,
\[ g(x) = \frac{2\Delta_2}{\ln \left( \frac{\psi_{\mathcal{E}_4}}{\psi_{\mathcal{E}_3}} \right)} = \frac{2\Delta_2 W(u_2, \ldots, u_k) W(u_1, \ldots, u_k, x^{l+1}e^{-\frac{x^2}{4}})}{W \left( W(u_1, \ldots, u_k) \right)} \]  
(4.48)
which is a solution of (4.33) and it is directly related with the corresponding PV solution through \( w(z) = 1 + \sqrt{z}/g(\sqrt{z}) \). Finally, the potentials (4.31) are
\[ V(x) = \frac{x^2}{8} + \frac{l(l+1)}{2x^2} + \ln W(u_1, \ldots, u_k)'' + \frac{\mathcal{E}_2 + \mathcal{E}_4 - 1}{2}. \]  
(4.49)
An example of this type of potentials, with a spectrum composed of a three-step ladder and an infinite one, is given in Figure 4.
5. Conclusions and outlook

In this paper we have presented a short overview of the polynomial Heisenberg algebras and explained how the methods of hsusy QM can be useful in this respect. So, we have shown that the higher order supersymmetric partners of the harmonic and radial oscillators provide the simplest non-trivial realizations of those deformed structures [11, 24]. We have analyzed as well the general systems ruled by the polynomial Heisenberg algebras when the differential ladder operators are of order one, two, three and four ($m = 0, 1, 2, 3$ respectively), and we have proved that the corresponding potentials involve Painlevé transcendentents of type IV and V in the last two cases (see also [7, 8, 13]). Although a part of the material here included is known in different contexts we thought it was appropriate to joint it in a self-contained work with a unified point of view. There are also some original results. Let us mention for instance the treatment of the whole section 4.4 devoted to the Painlevé V equation. Also, we have explored and generalized a reduction process using the $k$-susy QM applied to the
harmonic [24] and radial oscillators to construct a class of exact solutions to PIV and PV equations. The importance of our technique can be appreciated by comparing it with other more involved methods used to get them (see for instance Ref. [19]). The existence of other kind of solutions, although worth studying, is outside the scope of this paper (see however [25]).

Appendix

This appendix will be devoted to justify the formulas used in (4.25) and (4.46a). Let us consider the 2-susy transformation $B^+$ intertwining $H_0$ and $H_2$, $B^+ H_0 = H_2 B^+$, generated by two eigenfunctions $H_0 u_i = \epsilon_i u_i$, $i = 1, 2$. We can express $B^+$ in terms of $u_i$ in two ways, by means of Wronskians or through a consecutive action of 1-susy transformations:

$$B^+ \psi = \frac{W(u_1, u_2, \psi)}{W(u_1, u_2)}$$  \hspace{1cm} (A.1)

$$= A_{12}^+ A_{11}^+ \psi \equiv \left( \frac{d}{dx} - \frac{(A^+_1 u_2)^\prime}{A^+_1 u_2} \right) \left( \frac{d}{dx} - \frac{u_1}{u_1} \right) \psi \hspace{1cm} (A.2)$$

$$= A_{12}^+ A_{12}^+ \psi \equiv \left( \frac{d}{dx} - \frac{(A^+_2 u_1)^\prime}{A^+_2 u_1} \right) \left( \frac{d}{dx} - \frac{u_2}{u_2} \right) \psi.$$  \hspace{1cm} (A.3)

From any of the equations (A.1) or (A.2)–(A.3), we can check that $B^+ u_i = 0$, $i = 1, 2$. By taking the adjoint of $B^+$ we get the operator $B^-$ realizing the intertwining in the opposite way, $B^- H_2 = H_0 B^-$. If we choose equations (A.2)–(A.3) we get the following expressions for $B^-:

$$B^- = A_{12}^- A_{21}^- \equiv \left( \frac{d}{dx} + \frac{u_1}{u_1} \right) \left( \frac{d}{dx} + \frac{(A^+_1 u_2)^\prime}{A^+_1 u_2} \right)$$  \hspace{1cm} (A.4)

$$= A_{21}^- A_{12}^- \equiv \left( \frac{d}{dx} + \frac{u_2}{u_2} \right) \left( \frac{d}{dx} + \frac{(A^+_2 u_1)^\prime}{A^+_2 u_1} \right)$$  \hspace{1cm} (A.5)

Now, from (A.4)–(A.5) we find easily the eigenfunctions $H_2 \tilde{u}_i = \epsilon_i \tilde{u}_i$ which are annihilated by $B^-$. They are given by:

$$\tilde{u}_1 = \frac{1}{A_{12}^+ u_1} = \frac{u_2}{W(u_1, u_2)}, \hspace{0.5cm} \tilde{u}_2 = \frac{1}{A_{12}^+ u_2} = \frac{u_1}{W(u_1, u_2)}$$  \hspace{1cm} (A.6)

Repeating exactly the same arguments for an $n$–order susy transformation $B^+ H_0 = H_n B^+$, generated by $n$ eigenfunctions $H_0 u_i = \epsilon_i u_i$, $i = 1, \ldots, n$ such that $B^+ u_i = 0$, then the adjoint operator $B^-$ performing the opposite intertwining is characterized by $n$ eigenfunctions $H_n \tilde{u}_i = \epsilon_i \tilde{u}_i$, $i = 1, \ldots, n$ such that $B^- \tilde{u}_i = 0$ and such eigenfunctions are given by

$$\tilde{u}_1 = \frac{1}{A_{21}^+ u_1} = \frac{W(u_2, \ldots, u_n)}{W(u_1, \ldots, u_n)},$$

$$\vdots \hspace{1cm} \vdots \hspace{1cm} \vdots$$

$$\tilde{u}_n = \frac{1}{A_{12}^+ u_n} = \frac{W(u_1, \ldots, u_{n-1})}{W(u_1, \ldots, u_n)}.$$  \hspace{1cm} (A.7)

These ones are just the expressions appearing in (4.25) and (4.46b).
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