Dynamics as the preservation of a constant commutator

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Abstract

Some properties involving two operators with a constant commutator are derived. They include a definition of derivatives of operator functions, their conjugate spaces, and the associated translation generators. The cases of real functions, quantum coordinate and momentum operators, time and Liouville operator, and quantum time and energy operators, are analyzed within this formalism. This procedure allows the elucidation of the properties of time in classical and quantum mechanics.

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1. Introduction

Commutators appeared in non-relativistic quantum mechanics as a property between conjugate variables, leading to uncertainty relationships, evolution equations, and quantization rules. Another, not much used, property of constant commutators is the calculation of “derivatives” of operator functions of \( \hat{P} \) and \( \hat{Q} \) [1]. But the latter properties of constant commutators might have more applications, even outside the quantum realm.

On the other hand, there were some earlier indications that a derivative with respect to energy can be considered as a time operator in quantum mechanics [2]. Recently, quantum energy and time representations have been introduced and those results suggest that the time operator has the form \( i \hbar d/d \hat{H} \) [3].

In this Letter, we further develop the concept of derivatives of operators. We find that a commutation relationship like \( [\hat{A}, \hat{B}] = k \), where \( k \) is a complex constant, has many consequences relevant to physics, including time operators and classical and quantal evolution equations.

In Section 2, we start with the commutator between two operators, with a constant value, and derive its consequences regarding the derivative of functions of operators, eigenoperators of the commutator and the associated “translation operators”.

The discussion centers on “motion” in the space of eigenoperators of the commutator.

In Section 3, we apply the results of Section 2 in the space of real functions of \( x \). We obtain known facts of real functions of one variable, but this section is of help in understanding the use of the procedure we are introducing.

In Section 4, we revisit the case of non-commutation between the quantum coordinate and momentum operators. Our treatment deduces the known properties between these operators and the corresponding representations starting from the commutator relationship.

By taking other operators, also with a constant commutator, we get a new point of view about classical and quantum dynamics. In Section 5, we start with the commutator between the Liouville operator and the time associated to a single particle, written in terms of the coordinates and momenta, and derive evolution equations from it. It is shown that the evolution equation for a dynamical variable, or a probability density, is a consequence of the commutation relationship and that evolution takes place within the space of eigenoperators of the commutator.

The properties of quantum energy and time operators are developed in Section 6 starting from their commutator relationship. The advantage or our method is that it explains the parametric nature of the time variable, and also shows what is the appropriate time operator.

Finally, in Section 7 we present some concluding remarks.
2. Commutators and derivatives

Let \( \mathcal{V} \) be a linear vector space over the complex field. Let us consider two linear operators \( \hat{A} \) and \( \hat{B} \), acting on vectors in \( \mathcal{V} \), not necessarily self-adjoint, and a domain \( \mathcal{D} \) which is common to, and invariant under \( \hat{A}, \hat{B}, \hat{A}^{-1}, \hat{B}^{-1} \). Let us assume that \( \hat{A} \) and \( \hat{B} \) have the commutator

\[
[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = k,
\]

where \( k \) is a complex constant. There is no other restriction on \( \hat{A} \) and \( \hat{B} \) and all we will need are the properties of linear operators and of commutators. Since \( \hat{A} \) and \( \hat{B} \) are related as above, we call them conjugate.

Because the commutator is constant, its action on a vector \(|\psi\rangle \in \mathcal{D}\), including eigenvectors of \( \hat{A} \) or \( \hat{B} \), is the multiplication by \( k \)

\[
[\hat{A}, \hat{B}]\langle\psi| = k\langle\psi|.
\]

Note that if the operators commute, the commutator takes any vector to the null vector. If the commutator is not constant, \([\hat{A}, \hat{B}]\langle\psi| \) might have components orthogonal to \( |\psi\rangle \). Therefore, when the commutator is constant, the domain \( \mathcal{D} \) is invariant under the action of the commutator and it is not necessary to pay much attention to it.

A note apart is that, when we have an operation like \( \hat{B}\hat{A}|a\rangle \), where \(|a\rangle \) is an eigenfunction of \( \hat{A} \) with eigenvalue \( \alpha \), unless the commutator between \( \hat{A} \) and \( \hat{B} \) is zero, we cannot just move the eigenvalue \( \alpha \) to the left of \( \hat{B} \) but, instead we should use the commutation relationship in order to move it consistently because if \([\hat{A}, \hat{B}] = k \) and \( \alpha \) is treated as a constant that commutes with \( \hat{B} \), then

\[
\langle a|\hat{A}\hat{B}|a\rangle - \langle a|\hat{B}\hat{A}|a\rangle = \alpha (\langle a|\hat{B}|a\rangle - \langle a|\hat{B}|a\rangle) = 0,
\]

which is in contradiction with the assumption that \([\hat{A}, \hat{B}] = k \). The assumption that the eigenvalue of \( \hat{A} \) commutes with \( \hat{B} \) when these operators are conjugate has been used leading to misleading results [4].

Immediate consequences of (2.1) are

\[
[\hat{A}, \hat{B}^n] = kn\hat{B}^{n-1},
\]

\[
[\hat{A}^n, \hat{B}] = kn\hat{A}^{n-1},
\]

where \( n = 0, 1, \ldots \). These relations imply that at least one of \( \hat{A} \) and \( \hat{B} \) cannot be bounded [4]. In the case that \( \hat{A} \) and \( \hat{B} \) have an inverse, since \( \hat{B}[\hat{A}, \hat{B}^{-1}] = -k\hat{B}^{-1} \) and \( \hat{A}[\hat{A}^{-1}, \hat{B}] = -k\hat{A}^{-1} \), we also find that

\[
[\hat{A}, \hat{B}^{-n}] = -kn\hat{B}^{-n-1},
\]

\[
[\hat{A}^{-n}, \hat{B}] = -kn\hat{A}^{-n-1}.
\]

Then, for functions \( f(x) = \sum_{n=-\infty}^{\infty} f_n x^n \) and \( g(x) = \sum_{n=-\infty}^{\infty} g_n x^n \), that can be written in terms of a convergent power series, we can say that

\[
[\hat{A}, \hat{f}(\hat{B})] = k \sum_{n=-\infty}^{\infty} n f_n \hat{B}^{n-1},
\]

\[
[\hat{g}(\hat{A}), \hat{B}] = k \sum_{m=-\infty}^{\infty} m g_m \hat{A}^{m-1}.
\]

Also, accordingly with the properties of commutators,

\[
[\hat{A}, \hat{f}(\hat{B})\hat{g}(\hat{B})] = k \left\{ f(\hat{B}) \sum_{m=-\infty}^{\infty} g_m \hat{B}^{m-1} + \sum_{n=-\infty}^{\infty} f_n n \hat{B}^{n-1} \hat{g}(\hat{B}) \right\},
\]

\[
[\hat{f}(\hat{A})\hat{g}(\hat{A}), \hat{B}] = k \left\{ f(\hat{A}) \sum_{m=-\infty}^{\infty} g_m \hat{A}^{m-1} + \sum_{n=-\infty}^{\infty} f_n n \hat{A}^{n-1} \hat{g}(\hat{A}) \right\},
\]

\[
[\hat{A}, \hat{f}(\hat{A})\hat{g}(\hat{B})] = k f(\hat{A}) \sum_{m=-\infty}^{\infty} g_m \hat{B}^{m-1},
\]

\[
[\hat{f}(\hat{A})\hat{g}(\hat{B}), \hat{B}] = k \sum_{n=-\infty}^{\infty} f_n n \hat{A}^{n-1} \hat{g}(\hat{B}).
\]

These are the operator versions of the Leibnitz’ rule of differentiation of a product of functions.

Additional equalities are

\[
[\hat{A}^{-1}, \hat{B}^n] = -k \sum_{m=0}^{n-1} \hat{B}^{n-m-1} \hat{A}^{-2} \hat{B}^m,
\]

\[
[\hat{A}^n, \hat{B}^{-1}] = -k \sum_{m=0}^{n-1} \hat{A}^n \hat{B}^{-2} \hat{A}^m,
\]

\[
[\hat{A}^{-1}, \hat{B}^{-n}] = -\hat{B}^{-n}[\hat{A}^{-1}, \hat{B}^n] \hat{B}^{-n},
\]

\[
[\hat{A}^{-n}, \hat{B}^{-1}] = -\hat{A}^{-n}[\hat{A}^{-1}, \hat{B}^n] \hat{A}^{-n}.
\]

The above properties allow us to make the identifications

\[
k \frac{d}{d\hat{B}} \leftrightarrow [\hat{A}, \bullet],
\]

\[
k \frac{d}{d\hat{A}} \leftrightarrow [\bullet, \hat{B}],
\]

because \([\hat{A}, \bullet] \) and \([\bullet, \hat{B}] \) comply with the properties of a derivative of a function. With the \( \bullet \) in \([\hat{A}, \bullet] \) and in \([\bullet, \hat{B}] \) we mean that we should take the commutator with all that is at the right of it. We still have to be careful about the non-commutation of the operators. This conclusion is not valid if the commutator is zero or not constant. We will refer to \([\hat{A}, \bullet], kd/d\hat{B}, [\bullet, \hat{B}] \) and \( kd/d\hat{A} \) as superoperators (with a double hat on them) because they act on and changes an operator and because it has eigenoperators, instead of eigenfunctions. Superoperators have been in use for some time now in the field of statistical physics [5,6].

Since we are considering the commutator between operators, i.e. linear combination of operators, the properties of \( kd/d\hat{A} \) or \( kd/d\hat{B} \) are easy to find. If \( \hat{A} \) and \( \hat{B} \) are self-adjoint,

\[
(k \frac{d}{d\hat{B}} \hat{g}(\hat{B}))^\dagger = -k \frac{d}{d\hat{B}} \hat{g}(\hat{B}),
\]

\[
(k \frac{d}{d\hat{A}} \hat{g}(\hat{A}))^\dagger = -k \frac{d}{d\hat{A}} \hat{g}(\hat{A}).
\]
\[
\left( k \frac{d}{d \hat{A}} \hat{f}(\hat{A}) \right) = -k \frac{d}{d \hat{A}} \hat{f}(\hat{A}),
\]
\[ (2.21) \]

i.e. \( kd/d\hat{B} \) and \( kd/d\hat{A} \) are anti-self-adjoint.

The eigenoperators of the commutators are

\[
k \frac{d}{d \hat{B}} e^{a \hat{B}/k} = [\hat{A}, \bullet] e^{a \hat{B}/k} = a e^{a \hat{B}/k},
\]
\[ (2.22) \]

\[
-k \frac{d}{d \hat{A}} e^{-b \hat{A}/k} = -[\bullet, \hat{B}] e^{-b \hat{A}/k} = b e^{-b \hat{A}/k},
\]
\[ (2.23) \]

where \( a \) and \( b \) are parameters with the same units of \( \hat{A} \) and \( \hat{B} \), respectively, but they are not related to the eigenvalues of \( \hat{A} \) and \( \hat{B} \). These eigenoperators cause translations of \( \hat{A} \) and \( \hat{B} \), preserving their characteristics,

\[
e^{-a \hat{B}/k} \hat{A} e^{a \hat{B}/k} = \hat{A} + a,
\]
\[ (2.24) \]

\[
e^{b \hat{A}/k} \hat{B} e^{-b \hat{A}/k} = \hat{B} + b.
\]
\[ (2.25) \]

In particular, these translations preserve the commutator relationship, i.e. \( [\hat{A} + a, \hat{B} + b] = k \), since \( a \) and \( b \) are just parameters. With these results we can translate functions of operators,

\[
\hat{u}(\hat{A}, a) \equiv e^{-a \hat{B}/k} \hat{f}(\hat{A}) e^{a \hat{B}/k} = \hat{f}(\hat{A} + a),
\]
\[ (2.26) \]

\[
\hat{v}(\hat{B}, b) \equiv e^{b \hat{A}/k} \hat{g}(\hat{B}) e^{-b \hat{A}/k} = \hat{g}(\hat{B} + b),
\]
\[ (2.27) \]

with “evolution equations” given by

\[
-k \frac{\partial}{\partial a} \hat{u}(\hat{A}, a) = [\hat{B}, \hat{u}(\hat{A}, a)],
\]
\[ (2.28) \]

\[
k \frac{\partial}{\partial b} \hat{v}(\hat{B}, b) = [\hat{A}, \hat{v}(\hat{B}, b)].
\]
\[ (2.29) \]

The eigenoperators themselves,

\[
\hat{g}(a, \hat{B}) = e^{-a \hat{B}/k},
\]
\[ (2.30) \]

and

\[
\hat{h}(b, \hat{A}) = e^{b \hat{A}/k},
\]
\[ (2.31) \]

also have their differential equations,

\[
-k \frac{\partial}{\partial a} \hat{g}(a, \hat{B}) = \hat{B} \hat{g}(a, \hat{B}),
\]
\[ (2.32) \]

and

\[
k \frac{\partial}{\partial b} \hat{h}(b, \hat{A}) = \hat{A} \hat{h}(b, \hat{A}).
\]
\[ (2.33) \]

These equations suggest that a realization of the equalities in this section is obtained when we consider the space of functions of the eigenvalues \( a \) of \( [\hat{A}, \bullet] = kd/d\hat{B} \) where the form of the operator \( \hat{B} \) is \(-kd/da\) and the form of the operator \( \hat{A} \) is the multiplication by \( a \), because the commutator between \( a \) and \(-kd/da\) is also \( k \). Similarly, there is a space of functions of the eigenvalues \( b \) of \([\bullet, \hat{B}] \) in which the form of the operator \( \hat{A} \) is \( kd/db \) and the form of \( \hat{B} \) is \( b \). \( a \) (\( b \)) is not an eigenvalue of \( \hat{A} \) (\( \hat{B} \)) but the explicit form of the operator. We expect that the eigenvalues \( a \) of \( \hat{A} \) be a subset of the values \( \{a\} \), and similarly for \( \hat{B} \) and its eigenvalues \( b \). \( a \) (\( b \)) can coincide with the eigenvalues of \( \hat{A} \) (\( \hat{B} \)) when its spectrum is infinite and continuous. Furthermore, we can change from one space to the other by integral transforms of the form

\[
f(b) \propto \int da e^{ab/k} \hat{g}(a),
\]
\[ (2.34) \]

\[
g(a) \propto \int db e^{-ab/k} \hat{f}(b),
\]
\[ (2.35) \]

with the conditions that \( e^{ab/k} \hat{g}(a) \) and \( e^{-ab/k} \hat{f}(b) \) be periodic or zero at the boundaries. Then, in many cases, Laplace and Fourier transforms relate the conjugate spaces. The conjugate space can have physics significance some times, but most of the times it is an alternative space in which a solution of a differential equation can be found more easily.

Since the commutators \([\hat{A}, \hat{B}]\) and \([kd/d\hat{B}, \hat{B}]\) have the same values, for our purposes, \( \hat{A} \) can be replaced by \( kd/d\hat{B} \) and vice versa, when appropriate. The same applies for \( \hat{B} \) and \(-kd/d\hat{A} \).

The application of the eigenoperators to functions in the eigenspaces of \([\hat{A}, \bullet] \) and \([\bullet, \hat{B}] \) result in their translation,

\[
u(a, a') = e^{-a' \hat{B}/k} f(a) = e^{a' \hat{D}/da} f(a) = f(a + a'),
\]
\[ (2.36) \]

\[
v(b, b') = e^{b' \hat{A}/k} g(b) = g(b + b'),
\]
\[ (2.37) \]

where \( a' \) and \( b' \) are the eigenvalues of the commutators. Therefore, “evolution equations” for functions of \( a' \) and \( b' \) are

\[
-k \frac{\partial}{\partial a'} u(a, a') = \hat{B} u(a, a'),
\]
\[ (2.38) \]

\[
k \frac{\partial}{\partial b'} v(b, b') = \hat{A} v(b, b').
\]
\[ (2.39) \]

This ensures that, as \( a' \) and \( b' \) vary, the translated functions do not leave the eigenspaces of \([\hat{A}, \bullet] \) or \([\bullet, \hat{B}] \).

The meaning of the translation of \( \hat{A} \) is found when we apply \( \exp(-a' \hat{B}/k) \) to \( \exp(b' \hat{A}/k) g(0) \),

\[
g(a'; b') = e^{-a' \hat{B}/k} e^{b' \hat{A}/k} g(0) = e^{a' \hat{D}/da} e^{b' \hat{A}/k} g(0) = e^{b(\hat{A} + a')} g(0).
\]
\[ (2.40) \]

Then, the corresponding evolution equation is

\[
k \frac{d}{db'} g(a'; b') = (\hat{A} + a') g(a'; b'),
\]
\[ (2.41) \]

i.e., the application of \( e^{-a' \hat{B}/k} \) to \( e^{b' \hat{A}/k} g(0) \) results in another function which is solution of the evolution equation with a translated operator \( \hat{A} + a' \). Since \( a' \) is just a parameter, it can take any value. A similar situation occurs in the conjugate space, the evolution equation for

\[
f(a'; b') = e^{b' \hat{A}/k} e^{-a' \hat{B}/k} f(0) = e^{-a'(\hat{B} + b')/k} f(0),
\]
\[ (2.42) \]

is

\[
-k \frac{d}{da'} f(a'; b') = (\hat{B} + b') f(a'; b').
\]
\[ (2.43) \]
The application of $-kd/d\hat{A}$ to a linear combination of translated functions gives the amount of translation of the combination. If they are referred to the same origin of $b$,

$$-k\frac{d}{d\hat{A}} (e^{-b\hat{A}/k} f(0) + e^{-b\hat{A}/k} g(0)) = b(e^{-\hat{A}/k} f(0) + e^{-\hat{A}/k} g(0)).$$

(2.44)

The quantity with interpretation is $b$ and not its squared magnitude. It is not useful to form linear combinations of functions referred to different origins because the derivative operator will not give the time of the combination and, besides, will change the combination.

Additional results are

$$[[\hat{A}, \bullet], \hat{B}] = \left[ k \frac{d}{d\hat{B}}, \hat{B} \right] = k,$$

(2.45)

$$[\hat{A}, [\bullet, \hat{B}]] = \left[ \hat{A}, k \frac{d}{d\hat{A}} \right] = -k,$$

(2.46)

$$\hat{f} \left( k \frac{d}{d\hat{B}} \right) \hat{g}(k \frac{d}{d\hat{A}}) = k f' \left( k \frac{d}{d\hat{B}} \right) g \left( k \frac{d}{d\hat{A}} \right),$$

(2.47)

$$\hat{A}, \hat{f} \left( k \frac{d}{d\hat{A}} \right) = -k f' \left( k \frac{d}{d\hat{A}} \right) g' \left( k \frac{d}{d\hat{A}} \right) + k f \left( k \frac{d}{d\hat{B}} \right) g \left( k \frac{d}{d\hat{A}} \right),$$

(2.48)

$$[e^{(k/a)\hat{d}/d\hat{B}}, \hat{B}] = \frac{k}{a} e^{(k/a)\hat{d}/d\hat{A}},$$

(2.49)

$$[\hat{A}, e^{(k/b)\hat{d}/d\hat{A}}] = -\frac{k}{b} e^{(k/b)\hat{d}/d\hat{A}},$$

(2.50)

$$\hat{f} \left( k \frac{d}{d\hat{B}} \right) g \left( k \frac{d}{d\hat{A}} \right) = -k f' \left( k \frac{d}{d\hat{B}} \right) g \left( k \frac{d}{d\hat{A}} \right) + k f \left( k \frac{d}{d\hat{B}} \right) g' \left( k \frac{d}{d\hat{A}} \right).$$

(2.51)

These results indicate that

$$[\bullet, \hat{B}] \leftrightarrow k \frac{d}{d(k\hat{d}/d\hat{B})},$$

(2.53)

$$[\hat{A}, \bullet] \leftrightarrow -k \frac{d}{d(k\hat{d}/d\hat{A})}. $$

(2.54)

We can continue nesting commutators and obtaining formulae for derivatives with respect to the $n$th derivatives with respect to $\hat{A}$ or $\hat{B}$.

Given a set of operators $\{\hat{A}_m, \hat{B}_n\}$ such that $[\hat{A}_m, \hat{B}_n] = k$, the set of operators on $\mathcal{V}$ is divided into three subsets: (i) the set of operators that commute with $\hat{A}_m$ and $\hat{B}_n$, (ii) the set $\{\hat{A}_m, \hat{B}_n\}$, and (iii) the set of operators with a non-constant commutator with $\hat{A}_m$ and $\hat{B}_n$. It is only within the functions of $\hat{A}_m$ and $\hat{B}_n$ that we can define derivative of operators.

If $\hat{C}$ commutes with $\hat{A}_m$ and $\hat{B}_n$, then the spaces of functions $f(a)$ and $g(b)$ are equivalent to the spaces $\hat{C} f(a)$ and $\hat{C} g(b)$, i.e. they can also be translated with $e^{a\hat{B}/k}$ and $e^{b\hat{A}/k}$, respectively. Furthermore, $\hat{C}$ is not affected by the action of the evolution operators $e^{-a\hat{B}/k}$ and $e^{b\hat{A}/k}$, i.e. $\hat{C}$ is a conserved quantity.

For an operator $\hat{D}$ that do not commute with $\hat{A}$,

$$[\hat{A}, \hat{D}] = \hat{G},$$

(2.55)

where $\hat{G}$ is another operator, we find that

$$[\hat{A}, \hat{D}^n] = \sum_{m=0}^{n-1} \hat{D}^m \hat{G} \hat{D}^{n-1-m},$$

(2.56)

$$[\hat{A}, \hat{D}^{-n}] = \sum_{m=0}^{n-1} \hat{D}^{-m} [\hat{A}, \hat{D}^{-1}] \hat{D}^{-n+1+m},$$

(2.57)

$$[\hat{A}^n, \hat{D}] = \sum_{m=0}^{n-1} \hat{A}^m \hat{G} \hat{A}^{n-1-m},$$

(2.58)

$$[\hat{A}^{-n}, \hat{D}] = \sum_{m=0}^{n-1} \hat{A}^{-m} [\hat{A}^{-1}, \hat{D}] \hat{A}^{-n+1+m},$$

(2.59)

$$[\hat{A}^{-1}, \hat{D}^{-n}] = \sum_{m=0}^{n-1} \hat{D}^{-m} [\hat{A}^{-1}, \hat{D}^{-1}] \hat{D}^{-n+1+m},$$

(2.60)

$$[\hat{A}^{-n}, \hat{D}^{-1}] = \sum_{m=0}^{n-1} \hat{A}^{-m} [\hat{A}^{-1}, \hat{D}^{-1}] \hat{A}^{-n+1+m},$$

(2.61)

where $n > 0$, and

$$[\hat{A}, \hat{D}^{-1}] = -\hat{D}^{-1} \hat{G} \hat{D}^{-1},$$

(2.62)

$$[\hat{A}^{-1}, \hat{D}] = -\hat{A}^{-1} \hat{G} \hat{A}^{-1},$$

(2.63)

$$[\hat{A}^{-1}, \hat{D}^{-1}] = \hat{A}^{-1} \hat{D}^{-1} \hat{G} \hat{D}^{-1} \hat{A}^{-1}.$$ 

(2.64)

If it happens that some of the latter three commutators are constant, then we can define "integral" operators with respect to $\hat{D}$. If $[\hat{A}, \hat{D}^{-1}] = k_1$, then

$$[\hat{A}, \hat{D}^{-n}] = -k_1 n(n-1) \int \hat{D}^{-n} d\hat{D}.$$ 

(2.65)

If $[\hat{A}^{-1}, \hat{D}] = k_2$

$$[\hat{A}^{-n}, \hat{D}] = -k_2 n(n-1) \int \hat{A}^{-n} d\hat{A}.$$ 

(2.66)

And if $[\hat{A}^{-1}, \hat{D}^{-1}] = k_3$

$$[\hat{A}^{-1}, \hat{D}^{-n}] = k_3 n \hat{D}^{-n+1} = -k_3 n(n-1) \int \hat{D}^{-n} d\hat{D},$$ 

(2.67)

$$[\hat{A}^{-n}, \hat{D}^{-1}] = k_3 n \hat{A}^{-n+1} = -k_3 n(n-1) \int \hat{A}^{-n} d\hat{A}.$$ 

(2.68)

Then, a second set of eigenoperators of the commutator is

$$[\hat{A}, e^{a\hat{D}^{-1}/k_1}] = -\frac{a^2}{k_1^2} \int \hat{D}^{-2} e^{a\hat{D}^{-1}/k_1} d\hat{D}$$

$$= a e^{a\hat{D}^{-1}/k_1},$$

(2.69)

$$[e^{a\hat{A}^{-1}/k_2}, \hat{D}] = -\frac{a^2}{k_2} \int \hat{A}^{-2} e^{a\hat{A}^{-1}/k_2} d\hat{A}$$

$$= d e^{a\hat{A}^{-1}/k_2},$$

(2.70)
\[ [\hat{A}^{-1}, e^{\alpha \hat{D}^{-1}/k_3}] = \frac{\alpha^2}{k_3} \int \hat{D}^{-2} e^{\alpha \hat{D}^{-1}/k_3} d \hat{D} = \alpha e^{\alpha \hat{D}^{-1}/k_3}, \]  
(2.71)

\[ [e^{\alpha \hat{A}^{-1}/k_3}, \hat{D}^{-1}] = -\frac{d^2}{k_3} \int \hat{A}^{-2} e^{\alpha \hat{A}^{-1}/k_3} d \hat{A} = \alpha e^{\alpha \hat{A}^{-1}/k_3}, \]  
(2.72)

The procedure presented in this section is valid for any linear operators, being them Hermitian or not. In next section we will show that in fact, the commutator coincides with the derivative in the space of real functions. Later, we will use this property in order to define appropriate derivatives for some physical systems of interest. This is a generalization of the concept of derivative which can also be applied to operators.

3. Ordinary differentiation

We can see the usefulness of the procedure introduced in previous section by deriving some known facts of a particular, simple case, and then we can later analyze other, more interesting, cases.

In this section we consider the space of real functions of \( x \) with \( \hat{A} = d/dx \) and \( \hat{B} = x \). For these quantities we have that the commutator results in

\[ \left[ \frac{d}{dx} , x \right] = 1. \]  
(3.1)

Then \( k = 1 \) and the spaces of functions of \( x \) and functions of \( d/dx \) are conjugate. A first equality is

\[ \left[ \frac{d}{dx} , x \right] e^{mx} = me^{mx}, \]  
(3.2)

where \( m \) is a parameter with units of \( 1/x \). This equation shows that \( [d/dx, x] \) is another way of referring to the derivative operation, it is just an equality. We also get

\[ \left[ \frac{d}{dx} , x \right] e^{mx} = me^{mx}, \]  
(3.3)

This equation says that \( [d/dx, x] \) is a translation obeying the partial differential equation

\[ \frac{\partial}{\partial m} \left( \frac{d}{dx} , m \right) = \left[ \frac{d}{dx} , m \right] , x \right] . \]  
(3.5)

We can define a function that travels in the \( m \)-space of eigenvalues of the commutator by means of its eigenoperator,

\[ u(m, m') = e^{-m'x} f(m) = e^{m'd/m} f(m) = f(m + m') . \]  
(3.6)

A partial differential equation for this function is

\[ \frac{\partial}{\partial m} u(m, m') = \frac{\partial}{\partial m} u(m, m'). \]  
(3.7)

Other set of relationships begins with

\[ [\bullet , x] f \left( \frac{d}{dx} \right) = f' \left( \frac{d}{dx} \right) , \]  
(3.8)

This relationship indicates that \( [\bullet , x] \) inhibits a derivative operation, an “inverse operation” to \([d/dx, \bullet] \). Additionally,

\[ [\bullet , x] e^{s d/dx} = s e^{s d/dx} , \]  
(3.9)

where \( s \) is a parameter with the same units as \( x \). The eigenoperator allows us to define the translation

\[ g(x, s) = e^{s d/dx} f(x) e^{-s d/dx} = f(x + s) , \]  
(3.10)

with

\[ \frac{\partial}{\partial s} g(x, s) = \left[ \frac{\partial}{\partial x} , g(x, s) \right] . \]  
(3.11)

A function that we can define with the help of the eigenfunction of \([\bullet , x] \) is

\[ v(x, s') = e^{s' d/dx} g(x) = g(x + s') . \]  
(3.12)

This function is a solution of the partial differential equation

\[ \frac{\partial}{\partial s'} v(x, s') = \frac{\partial}{\partial x} v(x, s') . \]  
(3.13)

For \([d/dx, \bullet], x] \) we will obtain the same set of relationships as for \([d/dx, x], \) and so on.

The above are well-known facts, as is the fact that the \( x \) and \( d/dx \) spaces are related by the Laplace transform. The use of the conjugate space to \( x \) is as an aid when solving differential equations, for instance.

4. Coordinate and momentum operators

In this section, we derive some of the properties of the quantum operators \( \hat{P} \) and \( \hat{Q} \) and of the corresponding representations. Let \( \hat{A} = \hat{P} \) and \( \hat{B} = \hat{Q} \), then

\[ [\hat{P} , \hat{Q}] = -i\hbar , \]  
(4.1)

\[ [\hat{P} , \bullet] = -i\hbar \frac{\partial}{\partial \hat{Q}} , \]  
(4.2)

and

\[ [\bullet , \hat{Q}] = -i\hbar \frac{\partial}{\partial \hat{P}} . \]  
(4.3)

For a well-behaved function \( f \) evaluated at \( \hat{Q} \) we find that

\[ [\hat{P} , \bullet] f (\hat{Q}) = -i\hbar \frac{d \hat{f} (\hat{Q})}{d \hat{Q}} . \]  
(4.4)

Since the operators involved are self-adjoint and \(-i\hbar d/d\hat{Q} \) is a linear combination of these operators, \(-i\hbar d/d\hat{Q} \) is anti-self-adjoint.

A change in \( \hat{Q} \) is governed by \( \hat{P} \). An example of that is that \( \hat{P} \) causes a shift in the coordinates. Another relationship is

\[ [\hat{P} , \bullet] f (\hat{P}) = 0 . \]  
(4.5)
If we apply the commutator to the Hamiltonian operator, the above equations acquire physical meaning, namely

$$[\hat{P}, \bullet] \hat{H} = -i\hbar \frac{\partial \hat{H}}{\partial \hat{Q}}. \quad (4.6)$$

The eigenvalue equation for $[\hat{P}, \bullet]$ is

$$[\hat{P}, \bullet] e^{i\hat{P}/\hbar} = p e^{i\hat{P}/\hbar}, \quad (4.7)$$

where $p$ is a real parameter with units of momentum. With this, we can translate a function of $\hat{P}$

$$\hat{h}(\hat{P}, p) = e^{-ip\hat{Q}/\hbar} \hat{f}(\hat{P}) e^{ip\hat{Q}/\hbar} = \hat{f}(\hat{P} + p). \quad (4.8)$$

Then, a change in $p$ of $\hat{h}(\hat{P}, p)$ is governed by $\hat{Q}$:

$$i\hbar \frac{\partial}{\partial p} \hat{h}(\hat{P}, p) = [\hat{Q}, \hat{h}(\hat{P}, p)]. \quad (4.9)$$

The eigenoperator itself obeys the equation

$$i\hbar \frac{d}{dp} e^{-ip\hat{Q}/\hbar} = \hat{Q} e^{-ip\hat{Q}/\hbar}. \quad (4.10)$$

This indicates that, in the space of functions of $p$, the form of the operator $\hat{Q}$ is $i\hbar d\hat{Q}/dp$.

A function of $p$ is translated by applying the eigenoperator, i.e.

$$u(p, p') \equiv e^{-i\hat{P}/\hbar} f(p) = e^{p' d/\hbar} f(p) = f(p + p'), \quad (4.11)$$

where $p'$ is a real constant. The differential form of this property is

$$i\hbar \frac{\partial}{\partial p'} u(p, p') = \hat{Q} u(p, p'). \quad (4.12)$$

These are the known facts about the expression of the coordinate operator in momentum space.

On the other hand,

$$[\bullet, \hat{Q}] \hat{g}(\hat{P}) = -i\hbar \frac{dg(\hat{P})}{d\hat{P}}. \quad (4.13)$$

This equation also acquires physical meaning when applied to the Hamiltonian operator,

$$[\bullet, \hat{Q}] \hat{H} = -i\hbar \frac{\partial \hat{H}}{\partial \hat{P}}. \quad (4.14)$$

Now, $\exp(i\hat{Q}/\hbar)$ is an eigenoperator of $[\bullet, \hat{Q}]$,

$$[\bullet, \hat{Q}] e^{i\hat{Q}/\hbar} = q e^{i\hat{Q}/\hbar}. \quad (4.15)$$

The translation of $\hat{Q}$ is realized as

$$\hat{u}(\hat{Q}, q) \equiv e^{-i\hat{Q}/\hbar} \hat{f}(\hat{Q}) e^{i\hat{Q}/\hbar} = f(\hat{Q} + q), \quad (4.16)$$

and is dictated by the partial differential equation

$$i\hbar \frac{\partial}{\partial q} \hat{u}(\hat{Q}, q) = [\hat{P}, \hat{u}(\hat{Q}, q)]. \quad (4.17)$$

For the eigenoperator itself there is a differential equation

$$-i\hbar \frac{\partial}{\partial q} e^{i\hat{Q}/\hbar} = \hat{P} e^{i\hat{Q}/\hbar}. \quad (4.18)$$

These equations are telling us that in the space of functions of $q$, the form of the operator $\hat{P}$ is $-i\hbar d/dq$. A function in the $q$ space is translated when $e^{i\hat{Q}/\hbar}$ is used

$$v(q, q') \equiv e^{i\hat{Q}/\hbar} f(q) = e^{q' d/\hbar} f(q) = f(q + q'), \quad (4.19)$$

and the differential equation for this function is

$$-i\hbar \frac{\partial}{\partial q} v(q, q') = \hat{P} v(q, q'), \quad (4.20)$$

a well-known fact.

Note that the commutators $[\hat{P}, \hat{Q}], [\hat{P}, i\hbar \partial/\partial \hat{Q}]$ are also equal to $-i\hbar$ and then, similar results hold for $-i\hbar \partial/\partial \hat{Q}$ and $\hat{Q}$ or for $\hat{P}$ and $i\hbar \partial/\partial \hat{P}$, including uncertainty relationships.

In this case, the eigenvalues of $[\hat{P}, \bullet] (\hat{\bullet}, \hat{Q})$ coincide with the eigenvalues of $\hat{P} (\hat{Q})$. Also, as we know, the $q$ and $p$ spaces are related by the Fourier transform, and both of the conjugate spaces are of physical interest.

Most of the equalities in this sections are well known, but new results are found in the next two sections.

5. Classical Liouville operator

For classical systems, we consider the commutator between the one-dimensional Liouville operator

$$\hat{L}(\Gamma) = \frac{p}{m} \frac{\partial}{\partial q} + F(q) \frac{\partial}{\partial p}, \quad (5.1)$$

and the time variable $T(\Gamma)$, written in terms of $\Gamma = (p, q)$ after solving Hamilton’s equations of motion for a single particle. The commutator is

$$[\hat{L}(\Gamma), T(\Gamma)] f(\Gamma) = f'(\Gamma), \quad (5.2)$$

since

$$\hat{T}(\Gamma) T(\Gamma) = 1, \quad (5.3)$$

for a single particle, according to Hamilton’s equations of motion.

Therefore, $\hat{L}(\Gamma)$ and $T(\Gamma)$ are conjugate pairs because

$$[\hat{L}(\Gamma), T(\Gamma)] = 1, \quad (5.4)$$

and then

$$[\hat{L}(\Gamma), \bullet] f(T(\Gamma)) = f'(T(\Gamma)). \quad (5.5)$$

Actually, the above equation is the evolution equation for classical dynamical variables.

The eigenvalue equation for $[\hat{L}, \bullet]$ is

$$[\hat{L}(\Gamma), \bullet] e^{\lambda_0 T(\Gamma)} = \lambda_0 e^{\lambda_0 T(\Gamma)}, \quad (5.6)$$

where $\lambda_0$ is a real parameter, with units of $1/T$, that can take values from $-\infty$ to $\infty$. The translation of functions of $\hat{L}(\Gamma)$

$$\hat{u}(\hat{L}(\Gamma), \lambda_0) \equiv e^{-\lambda_0 T(\Gamma)} \hat{f}(\hat{L}(\Gamma)) e^{\lambda_0 T(\Gamma)}$$

$$= \hat{f}(\hat{L} + \lambda_0) \quad (5.7)$$
obeys the differential equation
\[ \frac{\partial}{\partial L_0} \hat{u}(\hat{L}(\Gamma), L_0) = \hat{u}(\hat{L}(\Gamma), L_0), T(\Gamma)). \]  
(5.8)

The eigenfunction
\[ g(L_0, T(\Gamma)) = e^{L_0T(\Gamma)} \]  
(5.9)
is the solution of the differential equation
\[ \frac{\partial}{\partial L_0} g(L_0, T(\Gamma)) = T(\Gamma) g(L_0, T(\Gamma)). \]  
(5.10)
Then, the form of \( T(\Gamma) \) in the \( L_0 \) space is \( \partial / \partial L_0 \). Functions of the variable \( L_0 \) can also be shifted within the eigenspace of \( [\hat{L}, \cdot] \)
\[ u(L_0, L') = e^{L'T(\Gamma)} f(L_0) = e^{L'd/dL_0} f(L_0) = f(L_0 + L'), \]  
(5.11)
and they solve the differential equation
\[ \frac{\partial}{\partial L'} u(L_0, L') = T(\Gamma) u(L_0, L'). \]  
(5.12)
The \( L_0 \) space does not seem to have physical meaning and it is a space in which the solution of a particular problem can be simpler.

On the other hand, we also have that,
\[ [\cdot, T(\Gamma)] f(\hat{L}(\Gamma)) = f'(\hat{L}(\Gamma)), \]  
(5.13)
in particular
\[ [\cdot, T(\Gamma)] e^{-t\hat{L}(\Gamma)} = -tie^{-t\hat{L}(\Gamma)}, \]  
(5.14)
where \( t \) is a parameter with units of time. The above equation indicates that \( \exp(-t\hat{L}(\Gamma)) \) is an eigenoperator of \( \hat{T} \equiv -d/d\hat{L}(\Gamma) \equiv [\cdot, T(\Gamma)] \) and cause a translation on time, i.e. it is the classical propagator. Then, the differential equation for the translation of a function of \( T(\Gamma), \)
\[ \hat{u}(T(\Gamma), t) \equiv e^{t\hat{L}(\Gamma)} f(T(\Gamma)) e^{-t\hat{L}(\Gamma)} = f(T(\Gamma) + t), \]  
(5.15)
is
\[ \frac{\partial}{\partial t} \hat{u}(T(\Gamma), t) = [\hat{L}(\Gamma), \hat{u}(T(\Gamma), t)]. \]  
(5.16)
For the translation of a function
\[ v(\Gamma, t) \equiv e^{-it\hat{L}(\Gamma)} f(\Gamma), \]  
(5.17)
the evolution equation is
\[ \frac{\partial}{\partial t} v(\Gamma, t) = -\hat{L}(\Gamma) v(\Gamma, t), \]  
(5.18)
which is the Liouville equation for a classical probability density.

When we apply the time superoperator to a linear combination of states gives the time of the combination, if they are referred to the same time origin,
\[ -\frac{d}{d\hat{L}} (e^{-it\hat{L}} f(0) + e^{-it\hat{L}} g(0)) \]
\[ = t(e^{-it\hat{L}} f(0) + e^{-it\hat{L}} g(0)). \]  
(5.19)
It is not useful to form linear combinations of functions referred to different origins because the time superoperator will not give the time of the combination and, besides, it will change the combination.

Note that there are three time-related quantities. The time \( T(\Gamma) \) for single particles (the arrival time), the quantity that appears in Hamilton’s equations; the time superoperator \( \hat{T} = -d/d\hat{L}(\Gamma) \); and the parameter \( t \), the eigenvalue of \( \hat{T} \), which is the quantity that appears in the Liouville equation.

The \( L_0 \) and \( t \) spaces are related by the Laplace transform, but we still have to find the physical meaning of the \( L_0 \) space, if there is any.

### 6. Quantum dynamics

Another interesting application is the conjugate pair formed by the quantum Hamiltonian and time operators. There are some studies in which a conjugate operator of \( \hat{H} \), the quantum Hamiltonian, has been constructed [7–11]. The conjugate operator \( \hat{T} \) is related to the quantization of the classical expression of time, in terms of coordinate and momentum, for single particles, or is the direct construction of the conjugate operator when there is neither coordinate nor momentum operators. That operator is useful when we are interested in determining arrival-time distributions. There is a large amount of literature on arrival-time distributions, and a few of them are the review article by Muga and Leavens [12] and the confined arrival time by Galapon [4,13–16], amongst other Refs. [7–11,17–50]. Some of the results in this section were derived by Egusquiza and Muga in [30].

So, let us assume that there is an operator \( \hat{T} \) such that
\[ [\hat{T}, \hat{H}] = i\hbar. \]  
(6.1)
An implication of this is
\[ [\hat{T}, \cdot] f(\hat{H}) = i\hbar \cdot \hat{f}(\hat{H}), \]  
(6.2)
i.e., \( [\hat{T}, \cdot] = i\hbar d/d\hat{H} \). We call this operator the time super operator \( \hat{T} \), and it is also conjugate to \( \hat{H} \). If \( \hat{T} \) is self-adjoint, then \( \hat{T} \) is anti-self-adjoint, \( \hat{T} = -\hat{T} \). Since \( [\hat{T}, \hat{H}] = i\hbar \), an uncertainty relationship is immediate, i.e. \( \Delta \hat{H} \Delta \hat{T} \geq \hbar / 2 \), where \( (\Delta \hat{A})^2 = \langle \psi | (\hat{A} - \langle \hat{A} \rangle)^2 | \psi \rangle \) for any operator \( \hat{A} \) and state \( | \psi \rangle \).

One of the eigenoperators of the commutator is \( \exp(-it\hat{H}/\hbar) \),
\[ \hat{T} e^{-it\hat{H}/\hbar} = [\hat{T}, \cdot] e^{-it\hat{H}/\hbar} = te^{-it\hat{H}/\hbar}, \]  
(6.3)
where \( t \) is a real parameter. This eigenoperator is the responsible for translation in time, i.e. evolution of operators and vectors. The operator function of \( \hat{T} \) and \( t \)
\[ \hat{O}(t, \hat{T}) \equiv e^{it\hat{H}/\hbar} \hat{O}(\hat{T}) e^{-it\hat{H}/\hbar} = \hat{O}(\hat{T} + t), \]  
(6.4)
is a solution of Heisenberg’s equation for the evolution of observables
\[ i\hbar \frac{\partial}{\partial t} \hat{O}(t, \hat{T}) = [\hat{O}(t, \hat{T}), \hat{H}]. \]  
(6.5)
We can have evolution of states by applying the eigenoperator to a ket
\[ |\psi(t)| \equiv e^{-i\hat{H}/\hbar} |\psi(0)|. \] (6.6)
This evolution is described through the Hamiltonian,
\[ i\hbar \frac{\partial}{\partial t} |\psi(t)| = \hat{H} |\psi(t)|. \] (6.7)
Here \( t \) is the eigenvalue of \([\hat{T}, \bullet]\) and is the evolution time, the variable that appears in the Schrödinger equation. Evolution is understood, then, as the scanning of the space of eigenoperators of \([\hat{T}, \bullet]\).

Now,
\[ [\bullet, \hat{H}] \hat{g}(\hat{T}) = i\hbar \hat{g}'(\hat{T}), \] (6.8)
and it follows that \([\bullet, \hat{H}]\) is \( i\hbar d/d\hat{T} \) and we call it \(-\hat{E}\), a superoperator with units of energy, but with continuous, infinite spectrum. Since \([\hat{T}, -i\hbar d/d\hat{T}] = i\hbar\), we also have an uncertainty relationship between \(\hat{T}\) and \(\hat{E}\).

\[ \exp(iE\hat{T}/\hbar) \] is the eigenoperator of \(\hat{E}\),
\[ \hat{E} e^{iE\hat{T}/\hbar} = -[\bullet, \hat{H}] e^{iE\hat{T}/\hbar} = E e^{iE\hat{T}/\hbar}, \] (6.9)
where \( E \) is a real parameter. It is clear now that the values of \( E \) do not correspond the values of the spectrum of \(\hat{H}\). The \( E \) space is conjugate to \( t \) but it does not correspond to the eigenspace of \(\hat{H}\), then the shift of \(\hat{H}\) can have any value and just corresponds to a shift of the zero energy level.

\( \hat{T} \) is the generator of a shifting of zero-energy level,
\[ \hat{u}(\hat{H}, E) \equiv e^{-iE\hat{T}/\hbar} \hat{f}(\hat{H}) e^{iE\hat{T}/\hbar} = \hat{f}(\hat{H} + E). \] (6.10)
This operator is a solution of the equation
\[ i\hbar \frac{d}{dE} \hat{u}(\hat{H}, E) = [\hat{T}, \hat{u}(\hat{H}, E)]. \] (6.11)
The application of the eigenoperator to a ket,
\[ |\psi(E)| \equiv e^{iE\hat{T}/\hbar} |\psi(0)| \] (6.12)
results in a translated ket with equation
\[ -i\hbar \frac{d}{dE} |\psi(E)| = \hat{T} |\psi(E)|. \] (6.13)
In the space of functions of \( E \), \( \hat{T} \) has the form \(-i\hbar d/dE\), and in the space of functions of \( t \), \(\hat{H}\) has the form \(i\hbar d/dt\). These spaces are related by the Fourier transform but the \( E \) space is not the space of eigenvalues of \(\hat{H}\). Quantum energy and time spaces were introduced in Ref. [3].

The meaning of a shift in energy level can be seen if we apply the translation operator to a ket at time \( t \)
\[ |\tilde{\psi}(t)| \equiv e^{iE\hat{T}/\hbar} e^{-i\hat{H}/\hbar} |\psi(0)| = e^{Ed/\hat{H}} e^{-i\hat{H}/\hbar} |\psi(0)| = e^{-i(\hat{H}+E)/\hbar} |\psi(0)|. \] (6.14)
The corresponding evolution equation is
\[ i\hbar \frac{\partial}{\partial t} |\tilde{\psi}(t)| = (\hat{H} + E) |\tilde{\psi}(t)|. \] (6.15)
Then, evolution takes place with the zero energy level increased by \( E \).

Here also, when we apply the time superoperator to a linear combination of states, it gives the time of the combination, if they are referred to the same time origin,
\[ i\hbar \frac{d}{d\hat{H}} \left( e^{-i\hat{H}/\hbar} |\psi(0)| + e^{-i\hat{H}/\hbar} |\phi(0)| \right) = t \left( e^{-i\hat{H}/\hbar} |\psi(0)| + e^{-i\hat{H}/\hbar} |\phi(0)| \right). \] (6.16)
The quantity with interpretation is \( t \) and not its squared magnitude. It is not useful to form linear combinations of functions referred to different time origins because it would not be clear what is the time the combination, and the combination will be modified.

Again, there are tree time-type quantities, usually \( \hat{T} \) is the operator obtained from the quantization of the classical expression for the time of a single particle, and then is related to the arrival-time [46]. The time superoperator \( \hat{T} \) returns the evolution time \( t \) of a wave packet or of a time-dependent operator, the quantity that appears in the Schrödinger and Heisenberg equations [3].

Finally, these energy and time spaces are related by a Fourier-type transform and the \( E \) space can be used to solve more easily a particular problem but does not correspond to the space of eigenvalues of \(\hat{H}\).

7. Remarks

We have shown that the preservation of the commutation relationship can be considered as a fundamental property that gives rise to the definition of time operators and evolution equations, among other things. Then, we have introduced another way of looking at classical and quantum dynamics, as a translation that preserves the value of a constant commutator.

By applying the method introduced in this Letter, one might be able to understand better other systems for which there is also a constant commutation relationship between two given operators.

Our treatment clearly shows why time is a parameter in classical and quantum theories. It labels the eigenoperators of the commutator involving the time for a single particle. This label is the quantity that appears in Liouville, Schrödinger and Heisenberg equations of motion.

Previously, the operator obtained by the quantization of the classical expression for the time of a single particle, written in terms of coordinate and momentum, was considered as the time operator [12,31], but it is related to the arrival-time. For the evolution of wave packets, we have shown that it is better to adopt \( i\hbar d/d\hat{H} \) as the time superoperator. This superoperator, when applied to the propagator, returns the time variable that appears in the Schrödinger equation, the evolution time.

The time superoperator does not perturb a time-dependent wave function because it always is one of its eigenfunctions. This is consistent with the idea of a clock variable that is embedded in classical and quantum equations of motion. A clock is an independent, noninteracting, device that is used as a refer-
ence in order to measure the changes of other systems and does not affect them.

In conclusion, if we are interested in changing a given quantity $\hat{a}$, what we can do is to identify an operator that has a constant commutator with $\hat{a}$, and then that will be the corresponding generator of translations of $\hat{a}$, and vice versa.

References