SPIN FOAM MODEL FOR 3D GRAVITY IN THE CONTINUUM

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In this talk I present a framework to take spin foam models to the continuum. The framework is presented taking 3d quantum gravity as an example and completing the program there. Theories in the continuum are defined using a generalization of the small lattice spacing limit of Lattice Gauge Theory (LGT) and the projective techniques used in Canonical Loop Quantization (CLQ). The existence of the theory in the continuum depends on the existence of this limit expressing the independence of the normalized partition function on the lattice in the limit of “small lattice spacing.”

1 The problem

The ultimate goal of this work is to quantize gauge theories in the absence of a background metric (gravity, gravity coupled to matter). Here I will deal only with 3d gravity, but seminal work of Plebanski could make an extension of this quantization program to 4d feasible.

As a guide to construct the framework a set of basic symmetries was selected. These symmetries are the following:

- *Internal gauge symmetry* (which motivates our choice of holonomies as primary variables as in LGT)
- *Diffeomorphism symmetry of General Relativity* (which forces us to use “all the embedded lattices”)

That is, I will define a theory in the continuum whose basic variables are the holonomies of the connection along the edges of embedded lattices. To organize the information from all the embedded lattices (or equivalently, to define a theory in the continuum) I will need to generalize the \( \lim_{a \to 0} \) of ordinary LGT. This generalization is the essence of kinematical CLQ. In this work we present an extension of this procedure to the covariant formalism. As may have been expected, this extension of the continuum limit of LGT turns out to be intimately related to Wilsonian renormalization (some renormalization ideas were recently applied to non embedded spin foams by Markopoulou and Oekl).

The underlying structure of spin foam models is that of Topological Quantum Field Theory. Instead of writing down the axioms, I will describe the motivation now and present the formal structure with the example of 3d quantum gravity. One assumes that at any “spatial” slice one can take measurements and completely characterize the state of the system. Thus a Hilbert space is assigned to every “spatial” slice. The system’s dynamics is specified by a propagator assigned to every manifold with boundary (spacetime) that takes states from “the initial” spatial slice to “the final” one. This assignment of propagators to chunks of spacetime and Hilbert spaces to “spatial slices” must satisfy all the properties expected from propagators and must be compatible with the covariance of General Relativity. In mathematical terms one demands the assignment to be a covariant functor from the category of cobordisms to the category of Hilbert spaces.
2 Embedded graphs and embedded polyhedra

In this section I will clarify the meaning of the term “all embedded lattices,” and explain how one can generalize the “lim_{a \to 0}” of ordinary LGT to this set of objects.

I should remark that in this work spaces and maps are piecewise linear. In the following, spacetime—a 3-manifold with boundary—will be denoted by \( M \), and the spatial slices—compact surfaces without boundary—will be denoted by \( \Sigma \).

The first step of this extension of loop quantization is to replace \( M \) by \( P(M) \)—the set of all polyhedra embedded into \( M \)—, and \( \Sigma \) by \( G(\Sigma) \)—the set of all graphs embedded in \( \Sigma \)—.

\( P(M) \) is the set of “all embedded spacetime lattices” and \( G(\Sigma) \) is the set of “all embedded Hamiltonian lattices.”

The key observation is that \( P(M) \) and \( G(\Sigma) \) are partially ordered by inclusion and directed. That is, a graph is bigger than all its subgraphs and given any two graphs there is a third graph that is bigger than both of them. The same happens for embedded polyhedra. Thus, \( \text{lim}_{\Sigma \to M} \) and \( \text{lim}_{X \to M} \) are meaningful. Another set of objects that is partially ordered by inclusion and directed is the set of regular lattices defined by a lattice spacing \( a_n = \frac{a_0}{2^n} \). The limits defined above reduce to the the usual \( \text{lim}_{a \to 0} \) when they are applied to this family.

3 Data from lattice gauge theory

Using standard methods of Hamiltonian lattice gauge theory (once the gauge group is fixed), one assigns a Hilbert space to every embedded lattice

\[
\Gamma \rightarrow C(\Gamma) = L^2(A/\mathcal{G}, d\mu_{\text{Haar}}).
\]

Similarly, the lattice regularization of the path integral assigns a partition function to every spacetime lattice. After a proper normalization, one must use a renormalization scheme to fix any free parameters present in \( \Omega^n_X \)

\[
X \rightarrow \Omega^n_X : C(\partial X) \to \mathbb{C}.
\]

3.1 Example: 3d Quantum Gravity (Turaev-Viro model)

3d gravity with Euclidean signature can be formulated as a SU(2) gauge theory.\(^7\) A spin foam quantization of this system\(^8\) lands in the Ponzano-Regge model.\(^9\) Here I will use its quantum group regularization developed by Turaev and Viro.\(^10\)

The Hilbert space assigned to every graph will be the q-deformation of the Hilbert space of SU(2) lattice gauge theory

\[
C(\Gamma) = \mathbb{C}[\text{adm}_{SU(2)}(\Gamma)]
\]

which is generated by the gauge invariant “functions” constructed using holonomies along the edges of \( \Gamma \) taken in different irreducible representations and contacted at the vertices using different intertwiners. A state of this spin network basis is characterized by a coloring of \( \Gamma \) with spins on its edges and with intertwiners on its vertices. The space of such admissible colorings is denoted by \( \text{adm}_{SU(2)}(\Gamma) \). In the non q-deformed case \( \text{adm}_{SU(2)}(\Gamma) \) generates the space \( L^2(A/\mathcal{G}, d\mu_{\text{Haar}}) \); the regularization just truncates the set of spins to be bounded by a maximum spin determined by \( q \).

The partition function depends on the fixed coloring on the boundary \( \alpha \), and a natural normalization is defined through dividing by the “vacuum state” defined by coloring all the
links with spin zero. The partition function is written as a state sum, where the states are colorings of the polyhedron (compatible with the boundary conditions) with spins on its faces and intertwiners on its edges

\[ \Omega_X^n(a) = \frac{\Omega_X(a)}{\Omega_X(j = 0)} \quad \Omega_X(a) = \sum_{\varphi} |X|_{\varphi}. \]

Here the weight \( |X|_{\varphi} \) is defined using the Turaev-Viro weight of a simple polyhedron \( X \) constructed as a deformation of \( X \), \( |X|_{\varphi} = |X_s|_{\varphi}^T \). It is a product of the weights assigned to the faces and the vertices; in addition, when the manifold has a boundary also one assigns weights to the edges of the boundary in order to have the gluing conditions expected from propagators. The weight assigned to faces with the topology of a plaquette is simply the \( q \)-analog of \( 2j + 1 \), the dimension of the space of states with \( J^2 = j(j + 1) \). And the weight assigned to the vertices is a \( 6 - j \) symbol (at every vertex of a simple polyhedron six faces meet). The properties of \( 6 - j \) symbols translate into invariances of the weights under certain deformations of the polyhedron or under refinements of it (for details see\(^{10,11}\)).

4 From lattices to the continuum

In this section I will describe how the LGT data from all the embedded lattices can be organized to define a single theory in the continuum. The key property will be compatibility with the partial ordering described in Sec. 2. This procedure for taking LGT to the continuum could be called Loop Quantization (LQ) because it extends the kinematics of CLQ, although it also prescribes a strategy to find the dynamics. In the diagram shown below, the row marked with asterisks (*) is the proposal of this work; the other row is given by CLQ.

\[
\begin{array}{ccc}
\Sigma & \xrightarrow{\text{LQ}} & C(\Sigma) \\
\ast M \ast & \xrightarrow{\text{LQ}} & \ast \Omega_M^n \ast \\
G(\Sigma) & \ast P(M) \ast & \text{LGT c. p. o.}
\end{array}
\]

Given an admissible coloring of a graph, one can extend it to any finer graph. One simply assigns the zero color to all the links not present in the original lattice. Thus, the Hilbert space assigned to a graph is contained in the Hilbert space assigned to any finer graph.

\[ \Gamma_1 \leq \Gamma_2 \Rightarrow C(\Gamma_1) \subset C(\Gamma_2) \]

Since given any collection of graphs one can find a graph finer than all of them, the Hilbert space assigned to this finer graph would contain all the spaces assigned to the original graphs. In this way, one can define the Hilbert space \( C(\Sigma) \) —the space assigned to the finest graph—

\[ C(\Gamma_1) \subset C(\Gamma_2), \ldots, C(\Sigma) \]

\[ C(\Sigma) \doteq \lim_{\Gamma \to \Sigma} C(\Gamma). \]

This is the Hilbert space that CLQ assigns to \( \Sigma \) (the continuum).
Now I describe the construction of the partition function of $M$ (the continuum). In a continuum limit the normalized partition function should be independent of the microscopic details. In the language of LGT, (in the limit of small $a$) it should be independent of the lattice spacing, $a$. Thus, if one adopted the lattice spacing as the length unit, all the physical quantities with dimensions of length (like the correlation length) should diverge. Here it does not make sense to take “the lattice spacing” as the length unit, but one can generalize the statement of “independence of the lattice spacing;” I will simply generalize $\lim_{a \to 0}$ to $\lim_{X \to M}$. Of course, if instead of all the embedded polyhedra $P(M)$ one considers the nested regular polyhedra defined by a lattice spacing $a_n = \frac{a_n}{2^n}$, the following $\lim_{X \to M} \Omega^n_X$ reduces to the usual $\lim_{a \to 0} \Omega^n(a, g(a))$.

For $\alpha \in C(\partial M)$ induced by $\alpha \in \text{adm}(\Gamma)$ one defines

$$\Omega^n_M(\alpha) \doteq \lim_{X \to M} \Omega^n_X(\alpha), \quad \partial X \geq \Gamma.$$ 

This is the heart of LQ, this covariant extension of loop quantization; it is a condition on the existence of the theory in the continuum. In the case of 3d Quantum Gravity this nontrivial result has been proven.\textsuperscript{11}

Remark:
The theory in the continuum was defined by postulating that the quotients of $q$-geometry($\partial M_0$) → $q$-geometry($\partial M_1$) transition amplitudes are measurable.

The work of Turaev and Viro is based on postulating that the transition amplitudes of $q$-geometry to $q$-geometry are measurable once a graph is fixed on the spacetime boundary. They prove that these transition amplitudes do not depend on the polyhedron interpolating between the components of the graph fixed in the boundary, but the transition amplitudes do depend on the graph fixed on the boundary. The theory developed in this work is projectively equivalent to the Turaev-Viro model.\textsuperscript{11}

5 Interpretation as a sum over quantum geometries

The definition of $C(\Sigma)$ was based on the possibility of extending admissible colorings of a graph to finer graphs. It will be useful to use the same property to define admissible colorings of the continuum, $\text{adm}(\Sigma)$.

$$\Gamma_1 \leq \Gamma_2 \leq \ldots \leq \Sigma \Rightarrow \text{adm}(\Gamma_1) \subset \text{adm}(\Gamma_2) \subset \ldots \subset \text{adm}(\Sigma)$$

Then, it is easy to see that

$$C[\text{adm}(\Sigma)] = C(\Sigma).$$

This new point of view can be used again in the definition of the partition function

$$\Omega_X(\alpha) = \sum_\varphi |X|_\varphi = \sum_\varphi |\varphi|$$

where $\partial \varphi = \alpha$, and the smallest polyhedron in which the coloring fits is contained in $X$, $X(\varphi) \leq X$. The last step is possible only if the weight depends only on the coloring and not on the carrying polyhedron, $|X|_\varphi = |X(\varphi)|_\varphi$.

Remarks:
1. $\Omega_X^n$ comes from a sum over histories (colorings) living in $M$. Later $\Omega_M^n$ is defined by taking the "regularizing box" $X' \to M$.

$|X|_\varphi = |X(\varphi)|_\varphi$ is only true for the anomalous weights of the T-V model. This compatibility of the weights and the embeddings gives physical reality to the notion of spacetime quantum geometry.\cite{8,2}

2. If $|X|_\varphi = |X(\varphi)|_\varphi$ and $X' \geq X$ the theory on $X'$ determines the theory on the coarser lattice, $X$. In the non q-deformed case this corresponds to integrating out extra degrees of freedom, "except for the fact that infinite factors may occur." The magic of the q-deformation is to make these factors finite.

3. $\varphi$ is not analogous to $[g]_{\text{Diff}}$ in the path integral quantization of general relativity. However, we can also realize $\Omega_X^n$ as a sum over classes of colorings and $|\varphi|$ is analogous to $[g]_{\text{Diff}}$.

6 Physical states and propagators

The structures defined up to now need to be refined if one wants the partition function to behave like a propagator. This refinement is standard in the construction of Topological Quantum Field Theories (TQFTs) and, in the language of canonical quantization of constrained systems, corresponds to considering the physical Hilbert space as kernel of the constraints.

In usual state sum models for TQFTs the refinement is described by the diagram

$$\begin{pmatrix}
C(\Gamma) \\
\Omega_X^n
\end{pmatrix} \longrightarrow \begin{pmatrix}
H(\Gamma) = C(\Gamma)/\ker(C(\Gamma)_{\times I}) \\
\Psi_X : H(\partial X_{t=0}) \to H(\partial X_{t=1})
\end{pmatrix}.$$ 

After this process, the resulting theory

1. is a TQFT.

2. is independent of chosen auxiliary structure $(\Gamma, X)$.

In the framework presented here

$$\begin{pmatrix}
C(\Sigma) \\
\Omega_M^n
\end{pmatrix} \longrightarrow \begin{pmatrix}
H(\Sigma) = \Omega_{\Sigma \times I}^n (C(\Sigma)) \subset C(\Sigma)^* \\
\Phi_M^n : H(\partial M_{t=0}) \to H(\partial M_{t=1})
\end{pmatrix}.$$ 

After this refinement is completed, the resulting theory

1. is a projective TQFT.

2. no auxiliary structure was singled-out during its construction.

3. when realized as a sum over classes of colorings, it is an implementation of the Reisenberger-Rovelli projection operator.\cite{12}

7 Action of the homeomorphism group

Since the theory is defined in the continuum, the relevant group of deformations has a clear action. A piecewise linear homeomorphism

$$f : M \to N$$
induces the faithful action
\[ U_f : C(\partial M) \to C(\partial N) \]
\[ \alpha \mapsto f^{-1\ast}(\alpha) \]

In addition, the action on the weights is covariant
\[ |X|_\varphi = |f(X)|_{f^{-1\ast}\varphi}, \quad \Omega_X(\alpha) = \Omega_{f(X)}(f^{-1\ast}\alpha) \]
\[ \Rightarrow \Omega^\varphi_M = \Omega^\varphi_{f(M)} \circ U_f. \]

At the level of the projective TQFT defined in the last section (in the physical Hilbert space), the theory provides a representation of the modular group/mapping class group
\[ f, g : \Sigma \to \Sigma \text{ isotopic} \Rightarrow U_f|_{H(\Sigma)} = U_g|_{H(\Sigma)} \]
\[ f \approx id \quad \Rightarrow \quad \Omega^\varphi_{f(M)} = \Omega^\varphi_M. \]

### 8 Other examples

The method of spin foam quantization naturally handles actions of the type \( S[A, B] = \int_M B \wedge F \) and constrained versions of the same actions. Among the theories that can be formulated as constrained BF theories are Yang-Mills and General Relativity (thanks to Plebanski’s work).\(^1\)

The outcome of spin foam quantization is a quantum theory in the form of a state sum model defined over a fixed polyhedron. Then one can apply the procedure described here to use this family of theories labeled by polyhedra to construct a theory in the continuum.

\[ \Omega_X(\alpha) = \sum_\varphi |X(\varphi)|_\varphi \]

\[ \Omega^\varphi_M(\alpha) = \lim_{X \to M} \Omega^\varphi_X(\alpha) \]

Here I list other theories that have already been treated successfully within the described framework:

1. The Crane-Yetter model,\(^{13}\) a quantization of 4d BF theory, can be treated successfully.
2. Also we can treat 2d YM.

For these theories the histories live in the continuum, \( |X|_\varphi = |X(\varphi)|_\varphi \) (for the anomalous version of the models).

There are more complex systems under study:

1. The natural candidate to be investigated is the Barrett-Crane model for 4d quantum gravity (with Euclidean signature). This model is a constrained double of the Crane-Yetter model.
2. A simpler candidate is a “quantum Husain-Kuchar model” that is also a constrained Crane-Yetter model.

The histories live in the continuum, but we have not proven the existence of the \( X \to M \) limit.
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References

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