Supersymmetric derivation of the hard core deuteron’s bound state*

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Abstract. A supersymmetric construction of potentials describing the hard core interaction of the neutron-proton system for low energies is proposed. The approach focuses on the bounded states of the Hulthén potential but the method applies on arbitrary short range radial potentials.

The neutron-proton system is the simplest composite nucleus which can exist in a single stable bound state: the deuteron. In absence of electrostatic forces the interaction is depicted by a short range central potential [1]. For example, the Gaussian and the square well potentials have deserved special attention because their mathematical simplicity. In contradistinction, the Yukawa potential has a deeper theoretical significance. It is close connected with the ‘exchange forces’ which are responsible of the binding between nucleons. Historically, the Yukawa potential is often quoted by the prediction of the π meson. For low energies, the Schrödinger equation has not analytical solutions for this potential. However, it can be conveniently approximated by using the Hulthén expression [2]:

\begin{equation}
V(r) = -\frac{\mathscr{V}_0}{e^{r/\alpha} - 1},
\end{equation}

where \( r \) is the distance between the nucleons, \( \alpha > 0 \) is a fixed length (the range of the potential) and \( \mathscr{V}_0 > 0 \) is a fixed energy (the strength of the potential). Thus, potential (1) can be considered as the stand in of the nuclide interaction [3]. The nucleons cannot approach each other closer than a certain distance, otherwise the nucleus would not show an almost constant nuclear density. Then we must resort to phenomenological arguments [4]. The usual approach is to modify the nuclear potential at small distances to be consistent with the experimental data. A simpler approach considers the so-called hard core model which is characterized by a short range infinite repulsion inside the attractive nuclear interaction [5]. In other words, a realistic potential must contain more than a radial term of the form (1): it must also include an infinitely high barrier term.

In this paper we investigate the supersymmetric (susy) derivation of the hard core deuteron’s bound state. We shall show that potential (1) and its susy partner \( \tilde{V}(r) \) are either isospectral (i.e., they share the same spectrum) or almost isospectral (the same spectrum but the ground state). The properly selected \( \tilde{V}(r) \) can represent the hard core neutron–proton interactions at low energies. We shall focus on the involved bound \( S \)-states, the study of the scattering states will be done elsewhere.
The Schrödinger equation of the potential (1) reduces to an eigenvalue equation for a particle in a one dimensional effective potential \( V_\ell(x) = \ell(\ell+1)/x^2 - V(x) \), where \( \ell \) is the azimuthal quantum number and \( x = r/\alpha \) is a dimensionless radial coordinate. We are looking for the S states of binding energy (i.e., negative energies and \( \ell = 0 \)), therefore the following equation holds
\[
\left[ \frac{d^2}{dx^2} + \frac{V_0}{e^x - 1} - k^2 \right] \psi(x) = 0,
\]
where \( V_0 = (2\mu \alpha^2/\hbar^2)\mathcal{V}_0 \), \( E = -k^2 = (2\mu \alpha^2/\hbar^2)\mathcal{V} \). \( \psi(x) \equiv xR(x) \); with \( R(x) \) the standard radial wavefunction and \( \mu \) the reduced mass. The conventional procedure of solution carries out the eigenfunctions [6]:
\[
\psi_n(x) = C_n e^{-k_\alpha x} (1 - e^{-x}) {}_2F_1(2k_n + 1 + n, 1 - n, 2k_n + 1; e^{-x}), \quad n = 1, 2, \ldots,
\]
with \( C_n \) a normalization constant
\[
C_n \equiv \alpha^{-3/2} \frac{\Gamma(n+2k)}{\Gamma(n+1)\Gamma(2k_n+1)} [2k_n(n+k_n)(n+2k_n)]^{1/2}.
\]
The corresponding eigenvalues are given by
\[
E_n = -k^2_n = -\left( \frac{V_0 - n^2}{2n} \right)^2, \quad V_0 > n^2, \quad n = 1, 2, \ldots.
\]
The problem involves two mutually dependent parameters: \( V_0 \) and \( E \). In practice, the energy eigenvalues may be given by experiment while the strength \( V_0 \) of the potential is to be determined. Therefore, equation (4) can be used to evaluate \( V_0 \) in terms of \( E_n \) (just as we are going to do). Hulthén analyzed the case for which equation (2) has a single eigenvalue [2, 4]. Thus, the Hulthén potential \( V_H(x) \) has a constrained strength (in our notation \( 1 < V_0 = V_H < 4 \)) and represents a first approach to the problem: a centrifugal barrier avoiding the overlapping of nucleons is missed.

We shall consider a potential (1) allowing the binding of only two states. The purpose of this convention will be apparent in the sequel. To describe a system with only two bounded energy levels, the strength of the potential has to be in the domain \( 4 < V_0 < 9 \). Providing these values of \( V_0 \) we get \(-16 < E_1 < -2.25 \) and \(-1.56 < E_2 < 0 \). In order to have an idea of the involved orders of magnitude we note that, although there is no a priori reason why \( \alpha \) should not be different for each of the systems described by \( V(x) \), we can take its numerical value as \( \alpha = 3.5 \). Such assertion is justified by the fact that the mean distance between nucleons (i.e., the size of the nucleus) is in the range of 2 or 4 \( \hat{A} \) [1]. Therefore, we get \( (\hbar^2/\alpha^2m_p) \simeq 4,6113 \text{ MeV} \). Here, we have assumed that the neutron and proton masses are equal to \( m_p \), hence \( 2\mu = m_p \). In this way the experimental value of the deuteron binding energy \( E_d = -2.22456614(41) \text{ MeV} \) [7] becomes, in dimensionless value, \( E_d \simeq -0.4825 \) \( (k_d \simeq 0.6946) \). Remark that this value is in the domain of the exited state \( E_2 \) rather than in the domain of the ground state \( E_1 \).

As regards the susy scheme we look for a superpotential \( w(x) \) solving the Riccati equation \(-w'(x) + w^2(x) = V(x) - \varepsilon \), where the factorization energy \( \varepsilon = -\kappa^2 \) is, in principle, any real number. If \( V_0 = 1 + 2\kappa \) one easily gets a particular solution
\[
w(x) = \kappa - \frac{1}{e^x - 1}, \quad \kappa > 0.
\]
As usual, the susy expression \( \tilde{V}(x) = V(x) + 2w'(x) \) gives the new potential:

\[
\tilde{V}(x) = -\frac{1 + 2\kappa}{e^x - 1} + \frac{1}{2\sinh^2(x/2)}
\]

which has been previously reported [8] and used to study susy phase-equivalent potentials [9] as well as to establish some interesting connections between susy and the variational method [10]. Now, the r.h.s. term of (6) behaves as a repulsive centrifugal barrier (with \( \ell = 1 \)) in the neighborhood of \( x = 0 \):

\[
\tilde{V}(x) \sim -\frac{1 + 2\kappa}{x} + \frac{2}{x^2}, \quad x << 1.
\]

For \( x > 1 \) the potential (6) rapidly becomes negligible (see Figure 1). Observe that the strength \( V_0 = 1 + 2\kappa \) plays the role of a coupling constant.

![Figure 1](image.png)

Figure 1. The two levels Hulthén potential \( V(r) \) and its susy partner \( \tilde{V}(r) \) with \( \alpha = 3\sqrt{3} \) and \( \gamma_0 \approx 31.2572 \) MeV. The deuteron’s binding energy \( E_d \approx -2.2245 \) MeV has been ticked on the frame. The traditional Hulthén potential \( V_H \) has a strength \( \gamma_H \approx 11.0173 \) MeV and the same value of \( \alpha \).

Let us remark the relevance of the centrifugal term in (7). Although it could be added \textit{by hand} to (1), there is no way to know how to manage the functions (3) in order to obtain analytical solutions for the patch-worked potential. Unlike this unpleasant situation, the susy approach allows to determine the eigenfunctions and eigenvalues of the potential \( \tilde{V}(x) \). First, let us factorize the Hamiltonians as follows

\[
H = A^\dagger A + \epsilon, \quad \tilde{H} = AA^\dagger + \epsilon,
\]

with \( A \equiv \frac{d}{dx} + w(x) \). Then, an intertwining relationship holds: \( \tilde{H}A = AH \). Now, if \( \psi(x) \) is an eigenfunction of \( H \) with eigenvalue \( E \), we have \( \tilde{H}(A\psi) = E(A\psi) \), \( A\psi \neq 0 \). Therefore, if \( \psi \in L^2(\mathbb{R}) \), we get the normalized eigenstate of \( \tilde{H} \):

\[
\tilde{\psi}(x) = (E - \epsilon)^{-1/2}A\psi(x).
\]

Now we fix \( \epsilon = E_1 = -k_1^2 \). Then the l.h.s. equation (8) leads to \( H\psi_1(x) = E_1\psi_1(x) \) and consequently \( A^\dagger A\psi_1(x) = 0 \). It is easy to check that \( A\psi_1(x) = 0 \) is a sufficient condition to get a
square integrable $\psi_1(x)$. On the other hand, equation (9) gives no a susy partner $\tilde{\psi}_1(x)$ of $\psi_1(x)$. Such a function should be obtained from $A^\dagger \tilde{\psi}_1 = 0$ for which we have $H \tilde{\psi}_1(x) = E_1 \tilde{\psi}_1(x)$. A simple calculation gives $\tilde{\psi}_1(x) \propto e^{k_1 x} \left(1 - e^{-x}\right)^{-1}$, which is obviously not square integrable in $[0, \infty)$ for $\kappa > 0$. Hence, the potential $\tilde{V}(x)$, with $V_0 = 1 + 2\kappa = 1 + 2k_1$, misses the ground state of $V(x)$ and admits only one bound state (see equation (9)): 

$$
\psi(x) \equiv (E_2 - E_1)^{-1/2} \left[d/dx \ln \psi_2(x) + w(x)\right] \psi_2(x),
$$

with eigenvalue

$$
E_2 = - \left(\frac{V_0 - 4}{4}\right)^2 = - \left(\frac{2k_1 - 3}{4}\right)^2.
$$

We go a steep further and impose that the numerical value of $E_3$ be determined by experiment and let it be equal to $E_d$, the dimensionless value of the deuteron binding energy, therefore

$$
k_1 = \frac{4k_d + 3}{2} \simeq 2.8892, \quad V_0 = 4k_d + 4 \simeq 6.7784
$$

which agrees with the previously stated domains of $V_0$, $E_1$ and $E_2$. Then, an initial potential (1), with range $\alpha = 3.7$ and strength $\gamma_0 \simeq 31.2572$ MeV, has a susy partner (6) allowing the binding of a single state $\tilde{\psi}(x)$ with an energy exactly equal to $\delta_d$. The related probability density has been plotted on Figure 2. Its features can be contrasted with those of the no core Hamiltonian $H$, labeled $|\psi_H(x)|^2$ in the same figure. Observe the displacement of $\tilde{\psi}$ with respect to $\psi_H$. An easy calculation shows that $\tilde{\psi}(x) \sim (1 - e^{-x}) \psi_H(x)$, hence near the origin $\tilde{\psi}$ goes to zero as $x^2$ and $\psi_H \sim x$. Thus, the probability to find the state $\tilde{\psi}(x)$ near the origin is minor than the probability of finding $\psi_H(x)$ at the same place.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{The deuteron’s ground state probability density for the traditional Hulthén case (free of hard core) $|\psi_H(x)|^2$ and the new one (including hard core) $|\tilde{\psi}(x)|^2$. The function $\tilde{\psi}(x)$ is given in (10) with the values displayed in Figure 1. Observe the displacement of $|\tilde{\psi}(x)|^2$ with respect to $|\psi_H(x)|^2$.}
\end{figure}

Let us discuss some of the various implications of our results. First, the susy procedure accomplishes the derivation of a short range potential $\tilde{V}(x)$ allowing only one bound state. This potential exhibits many of the qualitative features concerned with a hard core nuclear
potential. It behaves predominantly as an effective radial potential \( V_\ell(x) \), with \( \ell = 1 \), near the origin (see equation (7)) and goes negligible with the increasing of \( x \) in the region \( x > 1 \).

Second, it is clear from Figure 2 that there is a considerable probability of finding the two nucleons at distances larger than \( \alpha \). Therefore, the nuclear force connected with \( \tilde{V}(x) \) plays a relevant role because the neutron and the proton are outside each other’s range so much of the time. That is, of course, a well known feature in the behaviour of the deuteron’s ground state [1, 11]. It is interesting to pay attention on the smoothest of \( \tilde{\psi}(x) \). This is also a nodeless function, just as one might expect for the eigenfunction of a single stable bound state. Therefore, the wavefunction (10) corresponds to the hard core ground state eigenfunction of the deuteron.

Observe that the hard core hypothesis increases the strength (12) of potential (1) by \( V_0 \approx 3V_{0H} \), where \( V_{0H} = 2k_d + 1 \). In general, the nuclear forces are quite complicated and the assumption of a pure \(^3S\)-state does not suffice to explain the deuteron quadrupole moment, this makes necessary to introduce a tensor force [4]. However, it is well known that the Hulthén potential gives a good approximation for the binding energy in the terms discussed at the very beginning of the paper. On the other hand, as mentioned above, there are different ways to modify the nuclear potential in order to get a hard core model. The potential derived here could be tested by the nucleon-nucleon scattering approaches, where the hard core hypothesis seems to be compatible with the empirical data [5].

In general, if \( \varepsilon < E_1 \) then there is no square integrable eigenfunction of \( H \) annihilated by \( A \) and equation (9) means that every \( \psi_n(x) \) could have a susy partner \( \tilde{\psi}_n(x) \). A missing state fulfilling \( A^+\tilde{\psi}_n(x) = 0 \) could be also available if \( \tilde{\psi}_n \in L^2(\mathbb{R}^+) \). Some susy treatments, involving general solutions to the Riccati equation, lead to susy partner potentials sharing exactly the same spectra [12] (compare with [13]). Although such a method is successful to study a number of interesting potentials (see e.g. [14, 15]), its application on the Hulthén potential is out of the present scope.

In summary, the method developed in the paper relied on the assumption that it is possible to describe the interacting system of a proton plus a neutron by a Schrödinger equation. In this way, we were able to construct a hard core potential for the deuteron. Moreover, we gave a numerical estimation of the potential energy necessary to give the observed deuteron’s binding energy. It is remarkable that even very refined experiments at low energies do not suffice to determine more than the range and strength of the involved potential, leading the detailed shape completely indeterminate. I hope this work sheds a new light on the matter.

* A first version of this paper, written in 1999 when I was leaving the Theoretical Physics Department of the Universidad de Valladolid (Spain), was circulated among colleagues as a preprint (quant-ph/9912066) and obtained positive comments on the simplicity of the centrifugal term (6)–(7) while stimulated the discussion on its physical meaning. I hope the discussion will continue with the present version.

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